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A foundational theory of belief and belief change

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Abstract

We suggest a foundational representation for the notion of belief and belief change process based on the notion of an epistemic state and its associated Scott consequence relation. We study the basic belief change operations in this framework and compare the resulting theory with related approaches.

Keywords: Belief change; Belief; Epistemic states

1. Introduction

The problem how to revise our beliefs or theories in response to new information can be seen as one of the fundamental problems in logic, with immediate potential applications ranging from philosophy to databases and artificial intelligence (see [11] for a comprehensive survey of current approaches in this area and their applications).

It was realized quite early that the set of beliefs alone is insufficient for grounding a nontrivial belief revision process; some more structure need to be imposed on (or, more exactly, discerned from) our epistemic states in order to guide our decisions about what to retain and what to retract in the process of revising our beliefs with new data. The need for such a structure is especially evident if the belief set is considered to be a deductively closed theory. In fact, different approaches to the problem of belief change can be classified first of all on the basis of what structure they impose on belief sets.

The most influential approach, the AGM theory of belief change [1,8], has considered belief revision as a process of changing (deductively closed) belief sets based on some selection function choosing among their maximal subsets that are consistent with the new data. The need for such a preference mechanism, in addition to the belief set itself, stems

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from the fact that neither each such subset taken alone ("maxichoice" revision) nor their common part ("full meet" revision) are appropriate as reasonable solutions. The AGM theory has suggested a systematic approach to the problem in this framework both in terms of general rationality postulates it should satisfy and in describing semantic representations that conform to these postulates.

The AGM approach has often been criticized for taking deductively closed sets of propositions as representing belief states. It has been argued that most such propositions are purely derivative and arise simply as logical consequences of other, more basic, or explicit, beliefs we have. Consequently, such derived beliefs should be withdrawn when we remove beliefs that served as their justification. Speaking more generally, deductively closed belief sets do not account for the fact that some of our beliefs may be *reasons* for other beliefs (cf. similar remarks made by Gärdenfors in [8], Section 3.5). In particular, maximal subtheories of a belief set cannot be always considered as appropriate options for choice, since they may contain beliefs that have already lost their justification.

The above criticism has developed into an alternative approach according to which the corpus of beliefs is seen as generated by some (usually finite) set of propositions called its *base* (see, e.g., [6,13,17,23]). Changes of such base-generated belief sets are determined by changes in their underlying bases: deletion of a proposition should amount to deletion of some of the basic propositions that served as its ground, while a typical revision should amount to adding propositions to the base. Among other virtues, such a representation drastically reduces the set of available alternatives and consequently has definite computational advantages. It allows also to avoid some controversial features of the AGM theory, such as the famous postulate of recovery (see [22]).

The role of justification in belief acceptance, and the destiny of derived beliefs in performing belief change operations lie at the heart of the dispute between coherentist and foundationalist approaches to belief and knowledge (see, e.g., [9,18,28]). In this respect, the AGM theory is usually interpreted as a realization of the coherentist approach, while the theory of base change is seen as an embodiment of the foundational approach. The approach that will be suggested below can also be seen as belonging to the foundational trend, since it will presuppose that the structure of our epistemic state is determined ultimately by certain dependence relations holding among our beliefs. Our proposal, however, will be significantly different from the base-generation approach. The main reason for the difference lies in the necessity to comply with still another important requirement on adequate representations of epistemic states.

It would be natural to require that belief change operations should act as functions transforming our epistemic states into new epistemic states. To perform this role, however, they should satisfy the *principle of categorial matching* (see [11]): the representation of the epistemic state after a change should be of the same kind as that before the change. Apart from aesthetic considerations, compliance with this principle seems to be essential for an adequate representation of iterated belief change operations.

The (complicated) relations of the AGM approach with the principle of categorial matching are discussed in a companion paper [3]. Unfortunately, it was rightly argued by Gärdenfors and Rott in their review [11] that the base-generation approach also has a problem with the principle: though a base determines the new belief set resulting from a change, it does not always determine a natural new base after the change. For example, deleting $p \wedge q$ from a belief set generated by a base $\{p, q\}$ should remove either p or q. If the change has to be minimal, and the two basic propositions are independent (and equally preferred), we obtain two equally plausible sub-bases, $\{p\}$ and $\{q\}$, that satisfy all the requirements. In particular, it is reasonable to assume that the new, contracted belief set should coincide with what is common to these two options, namely, with $\mathrm{Th}(p \vee q)$. But unfortunately, there is no natural contracted base that would produce this result. Consequently (and contrary to the hope expressed in [14]), the base-generation approach also creates difficulties for representing iterated belief changes.

In fact, the above extremely simple example brings to surface the common problem for practically all current approaches to belief change. The problem can be described as follows: most accounts often force us to choose in situations we have no grounds for choice. True, imposing preferences on the available alternatives, as is done in many approaches, can sometimes "break ties" in such situations. However, it should be clear that such a solution cannot be comprehensive unless the alternatives are always totally ordered with respect to preference, and this is hard to achieve in most situations.

An interesting aspect of the above problem arises in the case of the AGM representation. The principle of informational economy that lies at the base of the AGM approach points out to maximal subtheories of the belief set as the only reasonable options for choice. However, the suggested "partial meet" construction takes intersection of such theories as the final solution. As was rightly argued by Isaac Levi in [20], such a solution violates the very principle of informational economy it was based on, since the intersection of maximal subtheories is already not maximal by itself, and hence is far from being optimal in the sense of preserving as much information as possible. It is important to observe also that a similar attempt to take intersections of preferred alternatives in the case of the base-generated representation turns out to be completely inadequate: thus, in the above example the intersection of the two preferred sub-bases $\{p\}$ and $\{q\}$ is simply empty. Speaking more generally, we will argue in the paper that taking intersections of the preferred alternatives may lead to a loss of information. This loss is not "seen" so far as we are seeking only to find belief sets produced by one-step changes, but it will be revealed in subsequent changes.

Our approach to the above problem will amount roughly to retaining all the preferred alternatives as parts of the new epistemic state. We claim that all these alternatives should remain vivid in the representation of the resulting state rather than transformed into a single "combined" solution. Otherwise we may lose important information. As we will see, this does not prevent us from determining a unique resulting set of beliefs; but we should remember more than that.

Our epistemic states will consist of a *set* of (admissible) belief sets of the agent. In addition, a suitable principle of belief acceptance will determine a unique set of beliefs associated with a given epistemic state. We will describe some natural belief change operations on epistemic states that will generate, in turn, the corresponding operations on associated belief sets. In this way our construction will satisfy the above-mentioned principle of categorial matching. In addition, these operations will help to explain how epistemic states can be produced starting from some "initial" states.

As we will discuss later, the main justification for our notion of an epistemic state can be found in current works on nonmonotonic reasoning that suggest a more fine-grained and plausible view of how our beliefs are formed. In addition, there are also at least two other theories of belief change that can be seen as sources of our account. The first is a theory of Fagin et al. [5] that suggested a representation of databases in terms of *flocks* of theories; this construction will be extensively discussed in what follows. The second one is a theory of *relational belief change* proposed by Lindström and Rabinowicz [21]; the latter suggested to treat belief revision as a relation holding between the initial belief set and a number of its candidate revisions. This theory is considered in the companion paper [3].

It turns out that epistemic states in our sense can be described syntactically using so-called Scott consequence relations [7,27]. The rules of these consequence relations will reflect basic *dependence relationships* holding among our beliefs. Due to the correspondence between epistemic states and Scott consequence relations, belief change operations on epistemic states will be representable also as operations on consequence relations.

We will see that base-generated belief states are representable as a special case of our epistemic states. Furthermore, we show in [3] that the notion of epistemic entrenchment is intimately connected with Scott consequence relations. Due to this connection, the AGM theory will be also subsumed by our approach, though it will acquire a different, foundationalist interpretation. Thus, the suggested framework provides, in effect, a common foundationalist “umbrella” for the current approaches to belief change.

2. Epistemic states and their belief sets

To begin with, let us look at our epistemic states as “black boxes”. As their primary function, they produce belief sets. This is not, however, all that is hidden in them, since otherwise we would not be able to make reasonable changes in response to new information. We can get some grip on this hidden structure, however, if we take it to be responsible for producing certain subsets of the belief set in case we are required to delete some of our beliefs. These contracted belief sets can be seen as results of decomposing the source belief set into “well-formed” parts generated by the underlying structure. Only such subsets can be considered as serious options for subsequent changes made to the epistemic state. Note that each epistemic state has its own “signature” in this respect, even if the associated belief set happens to be the same; each epistemic state singles out its own set of contracted belief sets. And it seems natural to try to take this set of “well-formed” subsets as giving a primary description of the source epistemic state, as something determining (or reflecting) its structure. Guided by this idea, we can reverse the picture and assign an epistemic reality to such belief sets as something that already were there before we made contractions. Then an epistemic state will look like an iceberg with the belief set as its “visible” top and the set of its well-formed or “admissible” subsets as constituting its body. In fact, we will give below a number of reasonable ways how such a structure could be created.

The above picture can be seen as giving a preliminary informal background for our understanding of epistemic states. However, the picture is still insufficient for resolving the second main problem mentioned in the introduction, namely the problem of choice. As we said, contracting the initial belief set may often leave us with a number of equally plausible alternatives. And instead of trying to transform these alternatives into one combined belief set, we suggest to keep all of them as parts of the new epistemic state. This means that our epistemic states will not always have a unique "top" belief set, but will have instead a number of "peaks", each representing a maximal plausible set of beliefs. Such epistemic states will be seen as directly representing epistemic situations involving a number of different plausible "views of the world".

The above mentioned epistemic situations are actually quite familiar in studies of nonmonotonic reasoning. In fact, the latter have enriched us with a new general approach to the notion of belief according to which our beliefs are formed with the help of *defaults*, or *expectations*, that we are willing to accept in the absence of evidences to the contrary. In many situations, however, different defaults may conflict with each other, and this usually gives us a number of maximal "coherent" subsets of defaults forming a basis for different plausible sets of beliefs we can hold in such situations. Extensions of default theories in Reiter's default logic [25] reflect this feature of default-based systems. An abductive system of Poole [24] has given a very simple and natural representation of such systems in terms of possible scenaria formed by admissible sets of defaults. Propositions that are inferred ("explained") by maximal admissible scenaria are seen as representing potential beliefs we can hold about the situation.

If we compare the above description with the "traditional" representation of belief states as deductively closed sets of beliefs, we immediately notice two main differences. First, potential default-generated beliefs may often be incompatible with each other, and even if they are not, they cannot be freely combined to produce new plausible beliefs. This means that such potential beliefs do not always form a deductively closed set. Second, a belief set derived from a given admissible set of defaults is not homogeneous; not all deletions or additions to this belief set constitute justifiable belief sets, but only those that are supported by some admissible sets of defaults. This means, in particular, that our potential beliefs are correlated, and that some of them serve as reasons, or justification, for others.

It is instructive in this respect to compare the above picture with a general theory of nonmonotonic inference that has been developed by Gärdenfors and Makinson [10]. According to this theory, nonmonotonic inferences are formed with the help of expectations that are added as auxiliary assumptions to the premises. The authors have shown, in particular, that Poole system can be considered as a special case of their framework with defaults serving as such expectations. Also, the authors have established a close correspondence between this expectation-based representation and their previous work on belief revision. However, this correspondence was shown to hold on the condition that the set of expectations is jointly consistent and deductively closed. These assumptions, however, are inappropriate for general Poole systems and mark actually the distinction between the AGM approach and the theory suggested in the present paper.

We are ready now to introduce our definition of an epistemic state.

Definition 2.1. An *epistemic state* $\mathcal{E}$ is a set of deductively closed theories. Each $u \in \mathcal{E}$ will be called an *admissible belief state of* $\mathcal{E}$.

Admissible belief states will represent potential belief sets that are actually envisaged by the agent; borrowing (and taking literally) Isaac Levi's terminology, they will represent what is considered by the agent as a *serious possibility*. The very restriction on such possibilities indicates that our potential beliefs are correlated. As we shall see, our epistemic states can indeed be described in terms of *dependence relationships* holding between propositions. As a special case, such dependence relations provide *justifications* for some of our beliefs in terms of other beliefs we have. In this way the framework of epistemic states will allows us to represent a general foundationalist approach to belief change. Still, our version of foundationalism will be very moderate (and in this way will avoid much of the criticism raised against it). Thus, the net of dependencies will not be assumed in general to have a well-founded acyclic structure starting with universally acceptable self-justified postulates. In fact, all we assume is that there is some dependence relation between propositions that should be respected in producing revised epistemic states.

To avoid possible complications at this stage, we will consider in the rest of this section only epistemic states consisting of a finite set of theories.

An epistemic state gives raise to a natural notion of belief acceptance that we are going to describe in the next definition. In fact, this definition can be immediately discerned from the above informal description of epistemic states as generated by certain sets of defaults or expectations.

Definition 2.2. A proposition will be said to be *believed* in an epistemic state $\mathcal{E}$ if it belongs to all (set-inclusion) maximal belief states from $\mathcal{E}$. The set of all propositions believed in $\mathcal{E}$ will be called a *belief set* of $\mathcal{E}$.

Thus, even if an epistemic state contains multiple maximal belief states, we can still believe in propositions that belong to all of them. It is important to observe that the AGM theory of belief change also presupposes the above criterion of belief acceptance with respect to propositions that "survive" a change. Thus, a proposition will belong to a contracted belief state if it belongs to all (selected) maximal sub-theories of the initial state that do not include the proposition being contracted. On our approach, this is how we should accept our beliefs from the very beginning.

Our principle of belief acceptance can be viewed as providing a foundationalist representation for the notion of belief. As in a purely coherentist account, the acceptability of beliefs is determined, ultimately, by their compatibility ("coherence") with other potential beliefs, rather than by their derivability from some basic postulates. Still, this compatibility is constrained by compliance with the dependence relationships embodied in the composition of the underlying epistemic state.

Belief change operations will be defined below as functions from epistemic states to epistemic states. Accordingly, though any epistemic state determines an associated belief set, changes of the latter will be determined by changes made to the underlying epistemic state.

The belief set associated with an epistemic state $\mathcal{E}$ is actually an intersection of maximal belief states from $\mathcal{E}$, and hence it does not always constitute an admissible belief state of $\mathcal{E}$ by itself. The latter will hold, however, in an ideal situation when an epistemic state contains a unique greatest belief state. We will call such epistemic states *determinate* in what follows. [2] Such states can be seen as subsuming the two "traditional" representations, discussed in the introduction.

The suggested notion of an epistemic state allows for a richer repertoire of *epistemic attitudes* that are representable in this framework as compared with "plain" belief sets (see [8] for discussing epistemic attitudes expressible in different representations of epistemic states). For example, a proposition that is not believed may still constitute an admissible belief with respect to an epistemic state if it belongs to at least one admissible belief state. A proposition A will be said to be *disbelieved* in an epistemic state $\mathcal{E}$ if there is no admissible belief state in $\mathcal{E}$ that includes A. This notion of disbelief highlights still another "foundational" feature of our epistemic states: a proposition A is disbelieved if it is not supported by any admissible belief state (even though it may be logically consistent with all of them). Notice, however, that for determinate epistemic states this notion of disbelief is reducible to a simple absence of belief.

Our epistemic framework allows to define also a distinction between believed and known propositions. We will say that a proposition is *known* (or "firmly believed", if you prefer) in an epistemic state $\mathcal{E}$ if it belongs to *all* belief states from $\mathcal{E}$. The latter, somewhat unusual, notion of knowledge [3] will play only a marginal role in the present study dealing with belief change. Still, it will be instructive in distinguishing changes in beliefs from changes in knowledge. In particular, our belief change operations will not change what is known to the agent.

2.1. Grounded, base-generated and default-generated epistemic states

As we said, a natural understanding of epistemic states consists in seeing them as generated by some set of default propositions. For example, a Poole system can be seen as an epistemic state with admissible belief sets corresponding to logical closures of allowable sets of defaults. Poole's preferred scenaria will correspond then to maximal admissible states. Epistemic states of this kind can be described as follows.

Let Th be an arbitrary classical consequence relation and $\mathbb{B}$ a set propositions. An epistemic state will be said to be *generated by a set of propositions* $\mathbb{B}$ if it coincides with the set of theories $\{\mathrm{Th}(B) \mid B \in \mathbb{B}\}$. Epistemic states that are generated in this way by some set of propositions will be called *grounded* ones. Note that in the case of a finite underlying language, all epistemic states will be grounded.

Grounded epistemic states are highly sensitive to the actual sets of propositions that generate them. For example, a state generated by a set $\{p, q\}$ will certainly be distinct from the state generated by a single proposition $\{p \wedge q\}$; though both determine the same belief set, their behavior in belief change will be different. Notice, however, that this does not make the notion of a grounded state syntax-dependent. Rather, it reflects an idea that some of the believed propositions are considered as reasons, or grounds, for others.

It turns out that grounded epistemic states are intimately connected with a representation of databases in terms of *flocks of theories* suggested by Fagin et al. in [5].

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[2] Thanks to the anonymous referee who suggested the name.

[3] Notice that our framework is thoroughly epistemic and does not involve any reference to objective truth.

By a *flock* we will mean an arbitrary set of sets of propositions $\mathcal{F}$. An epistemic state will be called *generated by a flock* $\mathcal{F}$ if it coincides with a set of theories

$$\{\mathrm{Th}(w) \mid w \subseteq u,\ \text{for some } u \in \mathcal{F}\}.$$

A flock will be called *finite* if it consists of a finite set of finite sets of propositions. To simplify the discussion, we will restrict our attention here to finite flocks. Then it is easy to show that the notion of a flock-generation is reducible to that of a grounded epistemic state. For a set of propositions w, we will denote by $w^{\wedge}$ the set of all conjunctions of propositions chosen from w. Now if $\mathcal{F}$ is a flock, we will denote by $\mathbb{B}_{\mathcal{F}}$ the set of propositions

$$\bigcup \{u^{\wedge} \mid u \in \mathcal{F}\}.$$

Then it is easy to show that an epistemic state generated by a finite flock $\mathcal{F}$ is generated, in effect, by the set of propositions $\mathbb{B}_{\mathcal{F}}$. Notice also that any set of propositions $\mathbb{B}$ can be identified with a flock $\mathcal{F}_{\mathbb{B}}$ consisting of all singular sets $\{B\}$, where $B \in \mathbb{B}$. This shows that the two notions are basically equivalent in the finite case. An extension of this correspondence to the infinite case will be given later in the paper.

As in the present paper, the flock representation of databases was used in [5] to define their contractions (deletions) and revisions (insertions). Our results, however, will be very different from that of [5]. The main difference stems from the fact that the authors of [5] adopted a "strongly foundationalist" criterion of belief acceptance and treated a proposition as valid with respect to a flock of theories if it belongs to *all* theories in the flock.

To demonstrate the difference with our understanding of belief, let us say that two flocks are *equivalent* if they generate the same epistemic state. Since all belief change operations that will be introduced in the paper are definable in terms of the associated epistemic states, equivalence of two flocks will imply that they will be equivalent under any future change, or *equivalent forever* in the terminology of [5]. Now, let $\mathcal{F}$ be a flock and u a set of propositions such that $u \subseteq w$, for some $w \in \mathcal{F}$. Then it is easy to see that flocks $\mathcal{F}$ and $\mathcal{F} \cup \{u\}$ give raise to the same set of generating propositions $\mathbb{B}_{\mathcal{F}}$, and consequently they are equivalent in the above sense. This shows, in effect, that inclusion-maximal elements of a flock are sufficient for determining the associated epistemic state. Consequently, only the latter will be relevant for determining subsequent changes of these states. According to the approach of [5], however, the validity of propositions with respect to a flock is determined, in effect, by *minimal* sets belonging to the flock. As we will see, this seemingly small distinction produces drastic differences in the behavior of the corresponding belief change operations, and makes the approach of [5] less plausible and far more complex than the one suggested below. For example, the following simple lemma gives a necessary and sufficient condition of equivalence for finite flocks (this problem was retained open in [5]).

Lemma 2.1. *Two finite flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent modulo* Th *iff each proposition from $\mathbb{B}_{\mathcal{F}_1}$ is* Th*-equivalent to a proposition from $\mathbb{B}_{\mathcal{F}_2}$ and vice versa.*

The proof follows immediately from the fact that in both cases the generated theories of an epistemic state will have the form $\mathrm{Th}(B)$, for some $B \in \mathbb{B}_{\mathcal{F}}$. The following example from [5] illustrates the above result. This example was used in [5] for showing that their sufficient criterion of equivalence is not a necessary condition.

Example 2.1. Consider the following two flocks

$$\mathcal{F}_1 = \big\{\{A, B, A \leftrightarrow B\}, \{A, A \leftrightarrow B\}, \{B, A \leftrightarrow B\}\big\},$$

$$\mathcal{F}_2 = \big\{\{A, A \leftrightarrow B\}, \{B, A \leftrightarrow B\}\big\}.$$

It can be easily checked that in both cases the set $\mathbb{B}_{\mathcal{F}}$ of generating propositions amounts to $\{A, B, A \leftrightarrow B\}$. Consequently, the two flocks are equivalent and reducible both to the flock $\{\{A, B, A \leftrightarrow B\}\}$.

Base-generated states

Following Hansson [14], we will see a base-generated belief state as a pair $(\mathbb{B}, \mathrm{Th})$, where $\mathbb{B}$ is a base and Th a classical consequence relation. To begin with, we identify such belief states with epistemic states that are generated by all subsets of the base.

Definition 2.3. An epistemic state $\mathcal{E}$ will be called *base-generated* by $\mathbb{B}$ with respect to Th if it coincides with a set of theories $\{\mathrm{Th}(u) \mid u \subseteq \mathbb{B}\}$. In this case $\mathbb{B}$ will be called a *base of* of $\mathcal{E}$.

As can be immediately seen, base-generation is a special case of flock-generation when the flock contains only one set of propositions. Consequently, any base-generated epistemic state will be grounded.

A set of propositions $\mathbb{B}$ will be called $\wedge$*-closed*, if, for any $A, B \in \mathbb{B}$, $A \wedge B$ also belongs to $\mathbb{B}$. Then it is easy to see that an epistemic state is base generated if and only if it is generated by a $\wedge$-closed set of propositions, namely by $\mathbb{B}^{\wedge}$.

Propositions from a base of a base-generated epistemic state can be freely combined in order to produce new admissible belief states. In particular, it is easy to see that $\mathrm{Th}(\mathbb{B})$ is the greatest theory, or belief set, of such an epistemic state, while $\mathrm{Th}(\emptyset)$ is its least theory. Thus, base-generation implies determination. Notice, however, that the base $\mathbb{B}$ has not been assumed to be classically consistent; if $\mathbb{B}$ is inconsistent, the generated belief set will be inconsistent. This case actually marks a difference between base-generation and default generation described below.

As we will see, base generated states are inappropriate as a general framework for belief change, since they are not invariant under some natural belief change operations. Still, they correspond to a most simple and common kind of epistemic situations.

Default-generated states

Common default-type systems of nonmonotonic reasoning impose various constraints on acceptable combinations of defaults. In a simplest case, however, the only restriction on such combinations amounts to preserving logical consistency. In this case defaults can be identified with *supernormal defaults* of Reiter's default logic [25], that is, with normal defaults without pre-requisites (see, e.g., [4]). Epistemic states generated by such defaults are quite similar to base-generated states, with the only distinction that only consistent sets of defaults are considered as admissible:

Definition 2.4. An epistemic state $\mathcal{E}$ will be called *default-generated* by $\mathbb{B}$ with respect to Th if it coincides with a set of theories

$$\{\mathrm{Th}(u) \mid u \subseteq \mathbb{B} \ \& \ u \text{ is consistent}\}.$$

In this case $\mathbb{B}$ will be called a *default base of* $\mathcal{E}$.

As can be seen, if the set $\mathbb{B}$ is not jointly consistent, the resulting default-generated epistemic state will not be determinate. Still, it will be flock-generated by maximal consistent subsets of defaults.

3. Supraclassical Scott consequence relations

In this section we describe a logical formalism that will provide a syntactic description for epistemic states.

We begin with a brief description of a general sequent calculus called Scott consequence relations [27]. We refer the reader to [2,7] for a more detailed exposition.

Scott consequence relations can be seen as a "symmetric" generalization of Tarski consequence relations. They involve rules or sequents of the form $a \vdash b$, where a and b are finite sets of propositions. An informal reading of such sequents in this paper will be

> *If all propositions from a are believed* (*or accepted*), *then one of the propositions from b should also be believed* (*accepted*).

A set of sequents is called a *Scott consequence relation* if it satisfies the following conditions:

$$A \vdash A; \qquad \text{(Reflexivity)}$$

$$\text{If } a \vdash b \text{ and } a \subseteq a',\ b \subseteq b', \text{ then } a' \vdash b'; \qquad \text{(Monotonicity)}$$

$$\frac{a \vdash b, A \qquad a, A \vdash b}{a \vdash b}. \qquad \text{(Cut)}$$

The notion of a sequent can be extended to include infinite sets in premises and conclusions by requiring that, for any sets of propositions u and v, $u \vdash v$ if and only if $a \vdash b$, for some finite $a \subseteq u, b \subseteq v$. This will secure also a *compactness* of the resulting consequence relation.

A set of propositions u is a *theory* of a Scott consequence relation $\vdash$ if $u \nvdash \overline{u}$, where $\overline{u}$ denotes the set-theoretic complement of u. The following lemma shows that theories can be seen as sets of propositions that are closed with respect to the sequents of a consequence relation.

Lemma 3.1. *u is a theory of a Scott consequence relation $\vdash$ if and only if $u \vdash a$ implies $u \cap a \neq \emptyset$, for any set of propositions a.*

Scott theories can be considered as "multiple-conclusion" analogues of theories of a Tarski consequence relation. Note, however, that Scott theories do not have all the usual properties of deductively closed theories. Most importantly, intersections of Scott theories are not in general theories. Nevertheless, due to compactness, we still have the following

Theorem 3.2.
(1) *If u is a theory and v a set of propositions such that $v \subseteq u$, then u contains a minimal theory u' including v.*
(2) *If u is a theory disjoint from v, then u is contained in a maximal theory u' disjoint from v.*

In particular, the above theorem implies that any theory of $\vdash$ is included in a maximal theory and contains a minimal theory of $\vdash$.

Any set of sets of propositions $\mathcal{E}$ generates a Scott consequence relation $\vdash_{\mathcal{E}}$ defined as follows:

$$a \vdash_{\mathcal{E}} b \equiv \text{ for any } u \in \mathcal{E}, \text{ if } a \subseteq u, \text{ then } b \cap u \neq \emptyset.$$

The basic result about Scott consequence relations, called Scott Completeness Theorem in [7], says that theories of a Scott consequence relation form the *canonical semantics* of the latter. Let $\mathcal{T}_{\vdash}$ be a set of all theories of a Scott consequence relation $\vdash$. Then we have

Representation Theorem. $\vdash = \vdash_{\mathcal{T}_{\vdash}}$.

An important consequence of this theorem is that Scott consequence relations are uniquely determined by their theories.

Note that a Tarski consequence relation is also determined by the set of its theories. However, the set of theories of any Tarski consequence relation is closed with respect to intersections, and consequently not all sets of theories constitute exactly the theories of some Tarski consequence relation. In addition, a Tarski consequence relation always admits a set of all propositions of the language as its (greatest) theory. In this respect, Scott consequence relations have a definite advantage for our subsequent applications in that they provide a more tight correspondence between sets of theories and consequence relations. In fact, the correspondence is exact for *finite* Scott consequence relations that have a finite number of theories: if $\vdash_{\mathcal{E}}$ is a consequence relation generated by a finite set of sets of propositions $\mathcal{E}$, then it can be shown that $\mathcal{E}$ will coincide with the set of all theories of $\vdash_{\mathcal{E}}$.

In a general case, however, if $\vdash_{\mathcal{E}}$ is generated by an arbitrary set $\mathcal{E}$, then any set from $\mathcal{E}$ will be a theory of $\vdash_{\mathcal{E}}$, but $\vdash_{\mathcal{E}}$ may have other theories as well. A set $\mathcal{E}$ will be called *compact*, if it coincides with the set of all theories of $\vdash_{\mathcal{E}}$. The above-mentioned feature of Scott consequence relation can now be formulated as saying that any finite set of theories is compact.

Supraclassicality

Let us consider now consequence relations in a language containing the usual classical connectives $\{\vee, \wedge, \neg, \rightarrow\}$. $\vDash$ will denote the classical entailment with respect to these connectives.

A usual Tarski consequence relation is called *supraclassical* if it satisfies the following condition:

If $a \vDash A$, then $a \vdash A$. (Supraclassicality)

Thus, a Tarski consequence relation is supraclassical, if it subsumes classical inference. A Tarski consequence relation will be called *classical* if it is supraclassical and satisfies the deduction theorem: $a, A \vdash B$ always implies $a \vdash A \to B$. Such consequence relations are commonly used in the literature on belief change for describing the underlying logic. Classical entailment $\vDash$ coincides with the least classical consequence relation. In what follows, we will always use Th as denoting a provability operator of some classical Tarski consequence relation, while Th_c will denote the provability operator corresponding to the classical entailment.

A Scott consequence relation will be called *supraclassical*, if it satisfies Supraclassicality. As can be seen, this rule requires all theories of a Scott consequence relation to be deductively closed sets.

Supraclassicality allows for replacement of classically equivalent formulas in premises and conclusions of sequents. In addition, it allows to replace sets of premises by their classical conjunctions: $a \vdash b$ will be equivalent to $\bigwedge a \vdash b$. Multiple conclusions, however, are not replaceable in this way by their disjunctions.

A supraclassical Scott consequence relation will be called *classically consistent* if all its theories are logically consistent. As can be easily checked, this holds if and only if the sequent $\bot \vdash$, where $\bot$ is an arbitrary classical contradiction, belongs to the consequence relation.

Singular Scott consequence relations

Notice that the set of sequents $a \vdash A$ with singular conclusions belonging to a Scott consequence relation $\vdash$ forms a Tarski consequence relation. We will call the latter the *Tarski subrelation of* $\vdash$ and will denote it by $\vdash^T$. It can be shown that theories of $\vdash^T$ are precisely intersections of theories of $\vdash$ (including the intersection of the empty set of theories, the latter being identified with the set of all propositions of the language).

Tarski consequence relations can also be seen as a special kind of Scott consequence relations. More exactly, there is a one-to-one correspondence between Tarski consequence relations and *singular* Scott consequence relations satisfying the following condition:

If $a \vdash b$, then $a \vdash B$, for some $B \in b$. (Singularity)

It is easy to see that singular Scott consequence relations are uniquely determined by their Tarski subrelations. In what follows, we will always identify Tarski consequence relations with their associated singular Scott consequence relations.

As we will see, Tarski consequence relations cannot serve as a general framework for belief change. Nevertheless, we will show that they are sufficient for determining revised belief sets resulting from singular, one-step changes. This fact will play an important role in establishing the connection of our approach with the AGM representation in [3].

There is an important weakening of singularity that will be already invariant with respect to our belief change operations. A Scott consequence relation will be called *prime* if it has a least theory. As can be easily shown, primeness amounts to the requirement that $\vdash a$ holds only if $\vdash A$, for some $A \in a$.

4. Consequence relations as representations of epistemic states

As follows from the above discussion, any epistemic state $\mathcal{E}$ generates a supraclassical Scott consequence relation $\vdash_{\mathcal{E}}$ such that belief states of $\mathcal{E}$ are theories of $\vdash_{\mathcal{E}}$. Also, the above Representation Theorem shows that any supraclassical Scott consequence relation can be seen as generated by some epistemic state. Thus, Scott consequence relations can be seen as a formalism giving a syntactic representation of epistemic states. Moreover, this will allow us to drop the restriction to finite epistemic states, made earlier, while preserving most of their required features, such as the existence of maximal theories. Thus, in what follows we will consider arbitrary *compact epistemic states*, that is, sets of theories $\mathcal{E}$ that coincide with the theories of $\vdash_{\mathcal{E}}$.

Notice that the consequence relation associated with an epistemic state should *not* be identified with a logic of the agent in some strong sense of the word, something belonging to her intellectual capabilities independently of the world and available information. Rather, it can be seen as expressing basic *dependence relations* holding between her beliefs, relations that are implicit in the structure of the underlying epistemic state. On this understanding, $A \vdash B$ can also be interpreted as saying "We cannot believe in A without believing in B" or, in short, "Belief in A *depends* on belief in B". The possibility of characterizing epistemic states using associated consequence relations can be rephrased now in the terminology of [8] as saying that an epistemic state is uniquely characterized by a set of epistemic dependence judgments that are acceptable with respect to it.

We provide below translations of the main notions defined earlier for epistemic states into the terminology of consequence relations.

Definition 4.1. A proposition A will be said to be *believed* in a consequence relation $\vdash$, if it belongs to all maximal theories of $\vdash$. The set of all believed propositions will be called a *belief set* of $\vdash$ and denoted by $\boldsymbol{K}_{\vdash}$.

Notice that for ordinary classical consequence relations belief in A collapses to provability of A in $\vdash$, that is, to $\vdash A$. Thus, only in the supraclassical case the definition provides us with a nontrivial concept.

Definition 4.2. A proposition A will be said to be *disbelieved* in a consequence relation if $A \vdash$ holds.

A proposition is disbelieved in a consequence relation $\vdash$ if it does not belong to any theory of $\vdash$. Notice that any sequent of the form $a \vdash$ is reducible to $\bigwedge a \vdash$. Consequently, the following lemma shows, in effect, that beliefs associated with a consequence relation are uniquely determined by its set of disbeliefs.

Lemma 4.1. *A is believed in $\vdash$ iff, for any a, $A, a \vdash$ implies $a \vdash$.*

Notice that the reverse dependence does not in general hold: two consequence relations having the same belief set may have different sets of disbeliefs.

A consequence relation $\vdash$ will be called *determinate* if it has a greatest theory. In this case the belief set $\boldsymbol{K}_{\vdash}$ will coincide with this greatest theory. The following lemma provides a syntactic characterization of determination.

Lemma 4.2. *A consequence relation is determinate iff, for any set a, $a \vdash$ implies $A \vdash$, for some $A \in a$.*

Tarski consequence relations and their epistemic states. As we already mentioned, a Tarski consequence relation can be considered as a special kind of Scott consequence relations. However, any Tarski consequence relation admits the set of all propositions as its greatest theory. This implies, in particular, that our notion of a belief set becomes trivialized for such consequence relations, since it will invariably coincide with the greatest (inconsistent) theory. Speaking more generally, the inconsistent maximal theory of a Tarski consequence no longer carries some nontrivial information about the associated epistemic state, but rather constitutes a "by-product" of its structural properties (more exactly, of the fact that it has no rules without conclusions). However, it turns out to be reasonable to identify an epistemic state associated with a Tarski consequence relation with the set of its *consistent* theories. In particular, a belief set of a Tarski consequence relation will coincide then with the intersection of its maximal consistent theories. So, we will introduce the following

Definition 4.3. A proposition A will be said to be *believed* in a supraclassical Tarski consequence relation $\vdash$, if it belongs to all maximal consistent theories of $\vdash$. The set of all believed propositions will be called a *belief set* of $\vdash$.

Notice that, according to the above definition, a belief set of a Tarski consequence relation will be always consistent.

If $\vdash$ is a supraclassical Tarski consequence relation, we will write $a \vdash$ as a shorthand for $a \vdash \boldsymbol{f}$. Now, having this (defined) notion at our disposal, we can use our earlier definitions of disbelief and determination in the context of Tarski consequence relations. It can be easily checked, in particular, that the above Lemmas 4.1 and 4.2 will remain valid in our new context.

4.1. Grounded consequence relations

We will describe now Scott consequence that correspond to grounded epistemic states.

Let Th be an arbitrary classical consequence relation and $\mathbb{B}$ a set propositions. A Scott consequence relation will be called *generated by a set of propositions* $\mathbb{B}$ relative to Th if it is generated by the set of theories $\{\mathrm{Th}(B) \mid B \in \mathbb{B}\}$.

As we will show now, there exists a purely internal characterization of such consequence relations. To begin with, for any supraclassical Scott consequence relation $\vdash$, we will denote by $\mathrm{Th}_{\vdash}$ the *classical* consequence relation defined as follows:

$$A \in \mathrm{Th}_{\vdash}(a) \quad \text{iff} \quad \vdash \bigwedge a \to A.$$

It can be shown that $\mathrm{Th}_{\vdash}$ is a greatest classical consequence relation included in $\vdash$. As we will see, it is this consequence relation that can be seen as the underlying logic of a given Scott consequence relation and its associated epistemic state.

The following notion defines an important class of propositions by their role in a Scott consequence relation.

Definition 4.4.

(1) If u is a theory of a Scott consequence relation $\vdash$ such that

$$u = \mathrm{Th}_{\vdash}(A),$$

for some proposition A, then u will be called a *ground theory* of $\vdash$, while A will be said to be a *ground proposition* of $\vdash$.

(2) A Scott consequence relation will be called *grounded* if it is generated by a set of its ground theories.

As the following result shows, grounded consequence relations provide an exact description of consequence relations generated by a set of propositions.

Theorem 4.3. *A supraclassical Scott consequence relation $\vdash$ is grounded iff it is generated by a set of propositions with respect to some classical consequence relation.*

As we have seen, there is an intimate connection between grounded epistemic states and flocks of theories from [5]. The following description extends this correspondence to the associated consequence relations (and infinite flocks).

A consequence relation will be called *generated by a flock $\mathcal{F}$* with respect to a classical consequence relation Th if it is generated by a set of theories

$$\{\mathrm{Th}(w) \mid w \subseteq u,\ \text{for some } u \in \mathcal{F}\}.$$

In this case $\vdash$ will be called *flock-generated* by $\mathcal{F}$.

As before, $w^{\wedge}$ will denote the set of all conjunctions of propositions from w. As a special case, $\emptyset^{\wedge}$ will denote a classical tautology $\top$.

The following result shows that flock-generated consequence relations coincide, in effect, with a special case of grounded consequence relations.

Theorem 4.4. *A consequence relation is flock-generated iff it is prime and grounded.*

The proof of Theorem 4.4 shows that any consequence relation generated by a flock $\mathcal{F}$ is generated, in effect, by a set of propositions

$$\mathbb{B}_{\mathcal{F}} = \bigcup \{u^{\wedge} \mid u \in \mathcal{F}\}.$$

4.2. Basic propositions and base-generated consequence relations

Similarly to base-generated epistemic states, base-generated consequence relations are consequence relations that are generated by all subsets of some base.

Definition 4.5. Let Th be a classical consequence relation and $\mathbb{B}$ a set propositions. A consequence relation will be called *base-generated* by a pair ($\mathbb{B}$, Th) if it is generated by a set of theories $\{\mathrm{Th}(u) \mid u \subseteq \mathbb{B}\}$. In this case $\mathbb{B}$ will be called a *base of* $\vdash$ (relative to Th).

As can be seen, base-generation is a special case of flock-generation when the flock contains only one set of propositions. Consequently, any base-generated consequence relation will already be grounded. More exactly, base-generation by $\mathbb{B}$ amounts to a generation by a set of propositions $\mathbb{B}^{\wedge}$.

A set of propositions $\mathbb{B}$ will be called $\wedge$*-closed*, if, for any $A, B \in \mathbb{B}$, $A \wedge B$ also belongs to $\mathbb{B}$. Then we immediately obtain

Corollary 4.5. $\vdash$ *is base-generated iff it is generated by a* $\wedge$*-closed set of propositions.*

It is easy to see that $\mathrm{Th}(\mathbb{B})$ is the greatest theory, or belief set, of a base-generated consequence relation, while $\mathrm{Th}(\emptyset)$ is its least theory. Thus, base-generation implies primeness and determination.

We will show now that propositions from a base can also be characterized in terms of their behavior in a consequence relation. For any set of propositions w, we will denote by $A \to w$ the set of implications $\{A \to B \mid B \in w\}$.

Definition 4.6. A proposition B will be called *basic* in a supraclassical Scott consequence relation if $a, B \vdash b$ always implies $a \vdash B \to b$.

As follows from the definition, basic propositions satisfy the deduction theorem, the only property needed to turn a supraclassical consequence relation into classical one. In other words, basic propositions are, in a sense, propositions that behave in a fully classical way in our supraclassical context. The following result provides a semantic characterization of such propositions.

Theorem 4.6. *A is a basic proposition of $\vdash$ iff, for any theory u of $\vdash$, $\mathrm{Th}_c(\{A\} \cup u)$ is also a theory of $\vdash$.*

The above theorem implies, in particular, that any maximal theory of $\vdash$ should contain A. Consequently, we have

Corollary 4.7. *Any basic proposition of $\vdash$ is believed in $\vdash$.*

We are going to show now that the set of all basic propositions forms a base of a base-generated consequence relation. We begin with demonstrating that any proposition belonging to some base of a base-generated consequence relation is a basic proposition.

Lemma 4.8. *If $\mathbb{B}$ is a base of $\vdash$, then any proposition from $\mathbb{B}^{\wedge}$ is a basic proposition of $\vdash$.*

The following result shows that any ground proposition of a base-generated consequence relation will already be basic.

Theorem 4.9. *A is a basic proposition of a base-generated consequence relation $\vdash$ iff A is ground in $\vdash$.*

Recall that any base-generated consequence relation is grounded. Moreover, the set of basic propositions can be easily shown to be $\wedge$-closed. Consequently, we immediately obtain the following

Corollary 4.10. *A consequence relation $\vdash$ is base-generated iff the set of its basic propositions forms a base of $\vdash$ (relative to* $\mathrm{Th}_{\vdash}$*).*

Thus, the set of all basic propositions of a base-generated consequence relation can be seen as its *canonical base*. This shows, in particular, that base-generated consequence relations provide an adequate and complete representation for base-generated belief states.

4.3. Default-generated consequence relations

Default-generated consequence relations are generated by all *consistent* subsets of some set of propositions.

Definition 4.7. Let Th be a classical consequence relation and $\mathbb{B}$ a set propositions. A consequence relation will be called *default-generated* by a pair $(\mathbb{B}, \mathrm{Th})$, if it is generated by a set of theories

$$\{\mathrm{Th}(u) \mid u \subseteq \mathbb{B} \ \&\ u \text{ is Th-consistent}\}.$$

$\mathbb{B}$ will then be called a *default base of* $\vdash$ (relative to Th) and its elements will be called *defaults of* $\vdash$.

Default-generation is another special case of flock-generation when the flock contains all maximal consistent subsets of some set of propositions. Consequently, any default-generated consequence relation will also be grounded. In addition, any such consequence relation will be classically consistent.

Default-generated consequence relations “almost coincide” with base-generated consequence relations. Thus, any base generated consequence relation will be either default-generated by itself (if it is classically consistent), or it can be transformed into such by adding a sequent $\bot \vdash$ to it (which amounts to deleting the inconsistent theory $\mathrm{Th}(\bot)$).

Similarly to basic propositions, defaults can also be characterized by their behavior in a consequence relation.

Definition 4.8. A proposition B will be called a *default* in $\vdash$ if $a, B \vdash b$ always implies $a \vdash B \to b, \neg B$.

Notice that if b above is non-empty, $a \vdash B \to b, \neg B$ is reducible to $a \vdash B \to b$. Consequently, in such cases defaults will behave in a "fully classical" way, exactly as basic propositions. In particular, the following semantic characterization of defaults can be easily obtained from the corresponding result for basic propositions.

Theorem 4.11. *A is a default proposition of $\vdash$ iff, for any theory u of $\vdash$, either $\neg A \in u$, or $\mathrm{Th}_c(\{A\} \cup u)$ is also a theory of $\vdash$.*

Note, however, that default propositions are not always believed in a consequence relation, since it may happen that $\neg A$ is also admissible. Still, if a consequence relation is default-generated, its defaults constitute a "canonical default base". This result can be obtained by an easy modification of the corresponding result for base-generation.

5. Contractions of epistemic states

In this section we are going to define a general contraction operation producing deletion of propositions from epistemic states. Our guiding idea will be that such an operation should result in disbelieving these propositions. And a most natural way to achieve this consists in a removal of all admissible belief states that include them.

Definition 5.1. Let $\mathcal{E}$ be an epistemic state. A *contraction* of $\mathcal{E}$ with respect to a set of propositions w is an epistemic state, denoted by $\mathcal{E} - w$, that is determined by the set of all admissible belief states from $\mathcal{E}$ that are disjoint from w.

A *singular* (or ordinary) contraction is a contraction by a singleton set $\{A\}$, otherwise it will be called a *multiple contraction* (cf. [12]).

A contraction of an epistemic state generates a certain change in its belief set. Clearly, if A belongs to the belief set, it will be removed from it as a result of its contraction. Notice, however, that our contraction operation will produce a nontrivial change even if A is not actually believed, but still constitutes a serious possibility (that is, when it is not disbelieved also).

As we will show formally later, our contraction operation has a pleasant property of *commutativity*, that is, iterated contractions can be performed in any order. Moreover, any finite multiple contraction can be achieved through a sequence of corresponding singular contractions. We will show also that contractions preserve the property of groundedness, or flock-generation. Nevertheless, our construction gives different results from those implied by the theory of flock-generated change suggested by Fagin et al. [5]. Let us consider the following example from their paper.

Example 5.1. An epistemic state, generated by a base $\{A, B, A \leftrightarrow B\}$, is contracted by a set $\{A, B\}$. The resulting epistemic state is determined by a base $\{A \leftrightarrow B\}$. This will also be a result of contracting the initial state by a sequence of contractions $\langle -A, -B \rangle$.

The construction of deletion in [5] gives in the latter case a flock $\{\emptyset, \{A \leftrightarrow B\}\}$ which is treated, in effect, as identical with its least element $\{\emptyset\}$, while in our construction it is reducible to $\{\{A \leftrightarrow B\}\}$. As a result, multiple deletions are not equivalent to sequences of singular deletions in [5].

It can be immediately seen that contractions do not preserve base-generation. Thus, returning to the example already discussed in Section 1, contracting $p \wedge q$ from an epistemic state generated by a base $\{p, q\}$ results in an epistemic state generated by a set of propositions $\{p, q\}$. As can be seen, this epistemic state is not base-generated (notice that this set is not $\wedge$-closed). In our present context, this is simply a sign that *base-generated epistemic states cannot serve as a general framework for belief change*. Actually, the same example shows also that even the set of all determinate epistemic states does not constitute an adequate framework for representing belief change.

Remark 5.1. Suppose for a moment that we allow disjunctions of base propositions as legitimate elements of the base (such a solution was studied by Hansson in [15]). Then in the above example we could take $\{p \vee q\}$ as representing the contracted base, and it would give us the desired new belief set. Still, we claim that such a solution will be inadequate. Notice that a subsequent deletion of p from this new base does not produce any effect, since p is already not believed in it. However, if we take the initial base $\{p, q\}$ and delete both $p \wedge q$ and p, then q will be believed in the resulting epistemic state. Speaking more generally, taking intersections of the preferred alternatives may lead to a loss of information. This loss is not "seen" so far as we are seeking to find belief sets produced by one-step changes, but it will be revealed in subsequent changes. As we will see, this is also the reason why Tarski consequence relations are inappropriate for representing epistemic states.

5.1. Contractions of consequence relations

We will give now a syntactic characterization of our contraction operation as an operation on consequence relations. As will be seen, such a reformulation will provide us with a useful tool for studying such changes.

Definition 5.2. A *contraction* of a consequence relation $\vdash$ with respect to a set of propositions w, denoted by $\vdash_{-w}$, is a least consequence relation including $\vdash$ and sequents $A \vdash$, for all $A \in w$.

The following lemma provides a direct description of the contracted consequence relation.

Lemma 5.1. *For any finite sets a and b, $a \vdash_{-w} b$ iff $a \vdash b, w$.*

Now we will show that contraction of a consequence relation corresponds to contraction of the associated epistemic state, that is, to removal of belief states containing the propositions being contracted.

Theorem 5.2. *If a consequence relation $\vdash$ is generated by a set of theories $\mathcal{T}$, $\vdash_{-w}$ is generated by the set of all theories from $\mathcal{T}$ that are disjoint from w.*

Among other things this theorem implies that contractions preserve finiteness and groundedness of consequence relations.

As can be seen from the definition of contraction, $\vdash_{-w}$ is a least extension of $\vdash$ in which all propositions from w are disbelieved. Moreover, the notion of disbelief can be defined in terms of contractions. The following simple result will play an important role in what follows.

Lemma 5.3. *A proposition A is disbelieved in $\vdash$ iff $\vdash_{-A}$ coincides with $\vdash$.*

For any consequence relation $\vdash$, we will denote by $\vdash_{\langle -A_1,\ldots,-A_n \rangle}$ the result of applying to $\vdash$ a sequence of contractions of A_i in that order. Our next result shows that a sequence of contractions always amounts to a multiple contraction.

Theorem 5.4. $\vdash_{\langle -A,-B \rangle} = \vdash_{-\{A,B\}}$.

This result immediately implies that contractions *commute*, that is,

$$\vdash_{\langle -A,-B \rangle} = \vdash_{\langle -B,-A \rangle} .$$

Consequently, the order of iterated contractions is inessential, and any finite multiple contraction can be achieved through a sequence of corresponding singular contractions.

The properties of contractions, described below, constitute their basic "structural" features. They will be used later to give an abstract characterization of our contraction operation.

To begin with, the following lemma shows that contractions of $A \wedge B$ and $\{A, B\}$ can be considered, respectively, as "joins" and "meets" of singular contractions with respect to A and B.

Lemma 5.5.
(1) $\vdash_{-(A\wedge B)} = \vdash_{-A} \bigcap \vdash_{-B}$;
(2) *$\vdash_{-\{A,B\}}$ is a least consequence relation containing both $\vdash_{-A}$ and $\vdash_{-B}$.*

The following result provides another "lattice" property of contractions.

Lemma 5.6. *If A is disbelieved in $\vdash_{-B}$, then $\vdash_{-(A\wedge B)}$ coincides with $\vdash_{-A}$.*

The following result shows that the Tarski subrelation of any Scott consequence relation is sufficient to determine belief sets resulting from singular contractions. This result is important in establishing the correspondence between our theory and the AGM framework in [3].

Lemma 5.7. *Let $\vdash^T$ be a Tarski subrelation of a Scott consequence relation $\vdash$. Then, for any proposition A, the belief set of $\vdash_{-A}$ coincides with the belief set of $\vdash^T_{-A}$.*

Though adequate for "one-step" singular contractions, Tarski subrelation will produce different iterative and multiple contractions. For example, if $\vdash$ is generated by a set of propositions $\{p, q\}$, then contracting both p and q will give $\emptyset$ as the set of generating propositions, while the same contraction for $\vdash^T$ will retain $p \vee q$ in the resulting belief set. Notice that this result goes against our intuitions, since $p \vee q$ is a purely derivative proposition that was present in the initial belief set solely due to the presence of p and q.

Finally, we want to justify an informal description of epistemic states, given at the beginning of Section 2. According to this description, epistemic states could be determined by their belief "outputs" resulting from contractions.

Two consequence relations will be called *contraction-equivalent* if they produce identical belief sets under any contraction. Then the following result shows, in effect, that any epistemic state is uniquely determined by such contracted belief sets.

Theorem 5.8. *Two consequence relations are contraction-equivalent only if they coincide.*

The above theorem allows to answer also a similar question posed by Fagin et al. in [5]. Two flocks will be called contraction-equivalent if they produce the same belief sets under any contraction. The above result implies then that this can hold only if they generate the same consequence relation. Consequently, two contraction-equivalent flocks will give the same results for any subsequent belief change operation. In the terminology of [5], they will be *equivalent forever*.

5.2. Abstract belief contraction systems

We will give now a description of the contraction operation in terms of a set of postulates describing its behavior. The starting point of our characterization is the following simple observation:

Lemma 5.9. *For any supraclassical consequence relation, $a \vdash b$ iff contraction of $\vdash$ with respect to $b \cup \{\bigwedge a\}$ coincides with contraction of $\vdash$ with respect to b.*

The above equivalence can be formulated as saying that $a \vdash b$ holds if and only if b can be disbelieved only if $\wedge a$ is also disbelieved. This result shows, in particular, that a consequence relation is uniquely determined by the set of its possible contractions. This will allows us to describe properties of contraction in an abstract way that does not presuppose the knowledge about what an epistemic state is. Our construction will remind the reader Gärdenfors' dynamic representation of classical logic, given in [8], where propositions are identified with functions on epistemic states.

An abstract *belief contraction system* is a pair $\langle \mathbb{E},]\,[\rangle$, where $\mathbb{E}$ is a set of objects called *epistemic states*, while $]\,[$ is a mapping from the set of propositions to the set of functions on $\mathbb{E}$. For any proposition A and an epistemic state $\mathcal{E} \in \mathbb{E}$, we will denote by $\mathcal{E}]A[$ the result of applying a function $]A[$ to $\mathcal{E}$; the epistemic state $\mathcal{E}]A[$ will be said to be obtained by *contracting* $\mathcal{E}$ with respect to A. We will use also the notation $\mathcal{E}]A, B[$ as a shorthand for $\mathcal{E}]A[]B[$.

A proposition A will be said to be *disbelieved* in an epistemic state $\mathcal{E}$, if $\mathcal{E}]A[= \mathcal{E}$. As can be seen from Lemma 5.3 above, this definition agrees with the definition of disbelief given earlier. In addition, Lemma 4.1 can be used to give an adequate definition of propositions believed in an epistemic state:

Definition 5.3. A proposition A will be said to be *believed* in an epistemic state $\mathcal{E} \in \mathbb{E}$ iff, for any proposition B, $A \wedge B$ is disbelieved in $\mathcal{E}$ only if B is disbelieved in $\mathcal{E}$.

Thus, the basic notions associated with epistemic states are definable in the above framework. Actually, we will show now that the formalism of belief contraction systems is equivalent in its expressive power to that of supraclassical consequence relations.

The following set of *rationality postulates* will be shown to provide a complete description of belief contraction systems.

(C1) If $\vDash A \leftrightarrow B$, then $\mathcal{E}]A[= \mathcal{E}]B[$ (logical equivalence)

(C2) $\mathcal{E}]A, A[= \mathcal{E}]A[$ (idempotence)

(C3) $\mathcal{E}]A, B[= \mathcal{E}]B, A[$ (commutativity)

(C4) If $\mathcal{E}]A, B[= \mathcal{E}]B[$, then $\mathcal{E}]A \wedge B[= \mathcal{E}]A[$ (inversion)

As can be seen, the first three postulates are not specific for contraction functions (they are valid, for example, for the above mentioned Gärdenfors's belief models). So the only nontrivial contraction postulate is (C4). It says that if A is disbelieved in $\mathcal{E}]B[$, then contractions of $\mathcal{E}$ with respect to $A \wedge B$ and A produce the same resulting epistemic state (see Lemma 5.6).

It can be shown that Inversion can be strengthened to the equivalence:

$$\mathcal{E}]A, B[= \mathcal{E}]B[\text{ if and only if } \mathcal{E}]A \wedge B[= \mathcal{E}]A[.$$

Taken together with other properties of contraction, the above equivalence shows, in effect, that the set of contraction functions forms a lattice in which $]A, B[$ and $]A \wedge B[$ play, respectively, the roles of meet and join of $]A[$ and $]B[$.

Due to commutativity, for any finite set of propositions a, we can safely denote by $\mathcal{E}]a[$ the result of contracting $\mathcal{E}$ with respect to all propositions in a in some order.

Now we are going to assign any epistemic state $\mathcal{E}$ a "canonical" consequence relation $\vdash^{\mathcal{E}}$ as follows:

$$a \vdash^{\mathcal{E}} b \text{ iff } \mathcal{E}]b \cup \{\wedge a\}[= \mathcal{E}]b[.$$

Our next result shows that the above correspondence provides an adequate translation of epistemic states into consequence relations which, in addition, "translates" abstract contraction functions into contractions of associated consequence relations.

Theorem 5.10. *For any epistemic state $\mathcal{E} \in \mathbb{E}$, $\vdash^{\mathcal{E}}$ is a supraclassical consequence relation. Moreover, for any proposition A, $\vdash^{\mathcal{E}]A[}$ coincides with the contraction of $\vdash^{\mathcal{E}}$ with respect to A.*

The above result shows that our specific representation for the notion of an epistemic state can also be obtained as a by-product of a certain reasonable behavior of such states under belief change.

5.3. Foundational belief contraction functions

As can be expected, a contraction of an epistemic state (or a consequence relation) produces a change in its belief set. Consequently, any epistemic state $\mathcal{E}$ generates a certain *belief contraction function*, namely the function assigning each proposition A a belief set of $\mathcal{E} - A$. The following definition gives a corresponding description for belief contraction functions generated by consequence relations.

Definition 5.4. A *belief contraction function* generated by a consequence relation $\vdash$ is a function (denoted by $\boldsymbol{K}_{\vdash} - A$) assigning each proposition A the belief set of $\vdash_{-A}$.

$\boldsymbol{K}_{\vdash} - A$ can be seen as a result of contracting the belief set $\boldsymbol{K}_{\vdash}$ with respect to A. Any such belief contraction function generated by some epistemic state will be called *foundational*.

A detailed description of such foundational belief contraction functions and their comparison with AGM contractions are given in [3]. We show there that such functions preserve much of the "rationality" behind AGM contractions. In particular, for determinate epistemic states the corresponding belief contraction will satisfy all the AGM postulates except recovery (K–6) and the last "connectedness" postulate (K–8).

The description in [3] is based on the following observation, which is an immediate consequence of Lemma 5.7 above.

Theorem 5.11. *If $\vdash^{T}$ is a Tarski subrelation of a Scott consequence relation $\vdash$, then $\vdash$ and $\vdash^{T}$ determine the same belief contraction function.*

As a result, the class of foundational belief contraction functions coincides with the class of belief contraction functions generated by (supraclassical) Tarski consequence relations. This property of generated belief change functions will hold also for the other two operations on epistemic states studied later, namely expansions and revisions. Consequently, so far as we are interested only in belief functions resulting from one-step changes (which is the main concern of the AGM theory), we can restrict our attention to functions generated by more familiar Tarski consequence relations.

Despite the above mentioned similarity between foundational belief contraction functions and AGM contractions, it can be shown that the former satisfy also the following characteristic property:

(K-8f) If $A \in \boldsymbol{K}-B$, then $\boldsymbol{K}-(A \wedge B) \subseteq \boldsymbol{K}-A$. (persistence)

This postulate does not follow even from the full list of the AGM postulates. So, our framework produces an alternative understanding of belief contractions that is not subsumed by the AGM theory. In fact, we show in [3] that foundational belief contractions constitute a "partial-order" generalization of *severe withdrawals* suggested recently by Rott and Pagnucco in [26]. Nevertheless, we show also that AGM contractions are representable in our framework via a certain combination of belief change operations on epistemic states.

5.4. Contraction of base-generated consequence relations

A study of contractions constitutes the main subject of base-oriented studies in belief change. This is due to the fact that base contractions do not satisfy recovery, and consequently they are not definable in terms of revisions. On all accounts, however, revisions are still definable in terms of contractions and expansions via Levi identity (see below). This makes contraction a more basic operation.

Base-generated epistemic states have many interesting specific features that do not hold for a general case. Thus, we will show now that any such state is uniquely determined by its associated belief contraction function. As a result, it is possible in principle to give an abstract characterization of "base-generated" contractions in terms of the properties of the associated belief contraction functions. However, as can be seen from Hansson's results in [15,17], no characterization of this kind is going to be simple. So, we will state here only the basic facts.

Our first result shows that the Tarski subrelation of a base-generated Scott consequence relation $\vdash$ is uniquely determined by its associated belief contraction function.

Theorem 5.12. *If $\vdash$ is a base-generated consequence relation, then $A \vdash C$ holds iff, for any proposition B, $A \in \boldsymbol{K}_{\vdash}-B$ implies $C \in \boldsymbol{K}_{\vdash}-B$.*

In fact, the above property can be seen as a "logical source" of Hansson's symmetry postulate [17]. An additional consequence of the above result that plays an important role in Hansson's characterization is that in the finite case any theory of a base-generated consequence relation has the form $\boldsymbol{K}-A$, for some A (this follows immediately from compactness of finite sets of theories).

Clearly, contractions cannot preserve all the original basic propositions. The following result shows that a basic proposition will remain to be a basic proposition of a contracted consequence relation if and only if it is still believed in it.

Theorem 5.13. *A basic proposition B of a consequence relation $\vdash$ will also be basic in $\vdash_{-w}$ iff $B \in \boldsymbol{K}_{\vdash}-w$.*

Finally, it turns out that basic propositions can also be "restored" from the associated belief contraction function. The following result shows that if arbitrary contractions of propositions of the form $B \to C$ preserve the belief in B, then B is a basic proposition.

Lemma 5.14. *B is a basic proposition of $\vdash$ iff, for any c, B is believed in $\vdash_{-(B \to c)}$.*

The above results demonstrate that, for base-generated states, the associated belief contraction functions are already sufficient for recovering their structure. This property, however, is highly specific for base-generated states and is not extendable even to finite grounded epistemic states.

6. Expansions

The general aim of the expansion operation, studied in this section, consists in adding propositions to our beliefs. There exists, however, a number of ways to achieve this in the framework of epistemic states. In most cases, the differences will influence only iterated belief change operations. Nevertheless, such differences between various kinds of expansions can be seen as one of the main varying parameters in our approach to belief change. A number of such operations of expansion are also studied in [3].

It might appear at first sight that adding a proposition A to an epistemic state should amount to elimination of all admissible belief sets of an epistemic state that do not contain A. Such an operation, however, would miss the intended meaning of adding A as a new *belief* rather than an established (known) fact. Note, for example, that if A is disbelieved, the result of such an expansion would be an empty epistemic state. Furthermore, such an expansion would be "ineliminable" in the sense that a subsequent attempt to contract A in this case will again result in an empty epistemic state.

Generally speaking, the differences between various kinds of expansions amount to determining the degree of firmness with which we will believe the added proposition. In this respect, the foundational approach adopted in this paper suggests that a strongest way of adding new belief consists in making it a *basic* proposition of the new epistemic state. This is captured by the notion of a basic expansion described below.

6.1. Basic expansions

Basic expansions introduce new propositions into an epistemic framework in such a way that they are treated as freely combined with other believed propositions. As we will see, if an epistemic state is base-generated, such an expansion amounts to an addition of a proposition to the base. This understanding of expansions is presupposed practically in all "base-oriented" approaches to belief change.

For any set of theories $\mathcal{T}$, we will denote by $\mathcal{T}_{+w}$ the set of theories

$$\{\mathrm{Th}_c(u \cup v) \mid u \in \mathcal{T} \text{ and } v \subseteq w\}.$$

Definition 6.1. Let $\mathcal{T}$ be the set of theories of a consequence relation $\vdash$. For any set of propositions w, the consequence relation determined by the set of theories $\mathcal{T}_{+w}$ will be called a *basic expansion* of $\vdash$ with respect to w and denoted by $\vdash_{+w}$.

Thus, basic expansion with respect to w is obtained by adding arbitrary subsets of w to all theories of a consequence relation and taking their logical closure. Notice that an expansion of a consequence relation results also in "expanding" the set of its theories, so the new consequence relation will actually be a subrelation of the source one. In other words, $a \vdash_{+w} b$ will always imply $a \vdash b$.

The following technical lemma shows that instead of the set of all theories, we can safely choose any set of theories generating a given consequence relation.

Lemma 6.1. *If $\vdash$ is generated by a set of theories $\mathcal{T}$, then $\vdash_{+w}$ is generated by the set $\mathcal{T}_{+w}$.*

The next theorem confirms that $\vdash_{+w}$ is indeed a consequence relation we are looking for.

Theorem 6.2. *$\vdash_{+w}$ is a greatest subrelation of $\vdash$ that makes all propositions from w basic ones.*

The following consequence of the above result shows that basic propositions of a consequence relation can be described in terms of basic expansions:

Corollary 6.3. *A is a basic proposition of $\vdash$ iff $\vdash_{+A}$ coincides with $\vdash$.*

For *singular* expansion with respect to a proposition A, $\vdash_{+A}$ is a consequence relation determined by the set

$$\mathcal{T} \cup \{\mathrm{Th}_c(u \cup \{A\}) \mid u \in \mathcal{T}\},$$

where $\mathcal{T}$ is a set of theories of $\vdash$. In other words, expansion of A is determined by adding A to each of the theories of a consequence relation and taking logical closure of the result. The following theorem provides a direct syntactic description of such expanded consequence relations.

Theorem 6.4. *$a \vdash_{+A} b$ iff $a \vdash b$ and $A \to a \vdash A \to b$.*

As for contractions, expansions of epistemic states generate corresponding operations on their belief sets.

Definition 6.2. For any consequence relation $\vdash$ and any set w, $\boldsymbol{K}_{\vdash}{+}w$ will denote the set of all propositions believed in $\vdash_{+w}$. The corresponding function will be called a *belief expansion function* generated by $\vdash$.

Recall that ordinary AGM expansions are defined directly via the equation $\boldsymbol{K}+A = \mathrm{Th}(\boldsymbol{K} \cup \{A\})$ (see [8]). The following simple example shows, however, that our generated belief expansion functions are distinct, in general, from AGM expansions.

Example 6.1. Let $\vdash$ be generated by two propositions $\{p, q\}$ with respect to the classical entailment. Then the belief set $\boldsymbol{K}_{\vdash}$ coincides with $\mathrm{Th}(p \vee q)$. Now, expanding $\vdash$ with $A = p \to q$ will result in a consequence relation having $\mathrm{Th}(p \wedge q)$ as its only maximal theory. Consequently, the latter will coincide with $\boldsymbol{K}_{\vdash}+A$. But $\mathrm{Th}(\boldsymbol{K}_{\vdash} \cup A)$ will coincide in this case with $\mathrm{Th}(q)$.

Still, it is easy to see that if $\vdash$ is determinate, $\boldsymbol{K}_{\vdash}+A$ will coincide with $\mathrm{Th}(\boldsymbol{K}_{\vdash} \cup \{A\})$. So in this case the underlying consequence relation will be of no importance in determining the resulting belief set, since it can be obtained by a direct addition of new propositions to the source belief set. However, identical expansions of belief sets can be produced by different basic expansions, and this will be revealed in subsequent contractions and revisions of the resulting belief set. Thus, adding either $A \wedge B$ or $\{A, B\}$ will produce the same belief set, but only the latter expansion makes both A and B basic propositions. Consequently, subsequent contraction of $A \wedge B$ will delete all new theories introduced by adding $A \wedge B$, but will usually retain theories of the form $\mathrm{Th}_c(u, A)$ and $\mathrm{Th}_c(u, B)$ introduced by adding $\{A, B\}$. In the principal case this would retain $A \vee B$ in the resulting belief set.

The following result shows that, just as for contractions, basic multiple expansions can be simulated by singular expansions.

Theorem 6.5. *For any consequence relation,* $\vdash_{+w \cup \{A\}} = \vdash_{\langle +w, +A \rangle}$.

Thus, basic multiple expansions are equivalent to sequences of singular expansions of their elements. Another consequence of the above result is that basic expansions commute, and hence they can be performed in any order.

An interesting fact about basic expansions is that an expansion with respect to A commutes with a contraction with respect to $\neg A$. In other words, a sequence of changes $\langle -\neg A, +A \rangle$ always produces the same change in epistemic states as the sequence $\langle +A, -\neg A \rangle$. As we will see later, in both cases we will obtain a revision of an epistemic state with respect to A.

In dealing with infinite epistemic states and their associated consequence relations we should always check that our operations on epistemic states exactly correspond to the counterpart operations on consequence relations. In our present case, we need to show that if $\mathcal{T}$ is a set of all theories of $\vdash$, its expansion $\vdash_{+A}$ does not have new theories beyond $\mathcal{T}_{+A}$:

Theorem 6.6. *The set of all theories of* $\vdash_{+A}$ *coincides with* $\mathcal{T}_{+A}$.

The following consequence of the above result shows that expansions preserve basic propositions.

Corollary 6.7. *Any basic proposition of* $\vdash$ *remains a basic proposition of* $\vdash_{+A}$.

The following example shows that expansions may transform a singular Scott consequence relation into a non-singular one.

Example 6.2. It is easy to check that expanding the classical entailment $\vDash$ with atomic propositions p and q gives us a consequence relation $\vdash$ generated by a base $\{p, q\}$. This consequence relation is not singular: we have $p \vee q \vdash p, q$, though neither $p \vee q \vdash p$, nor $p \vee q \vdash q$.

The above example shows once more why ordinary Tarski consequence relations are inadequate for representing changes of epistemic states. Moreover, if an expansion of a Scott consequence relation produces an inconsistent belief set, it will be distinct from the (consistent) belief set generated by expanding its Tarski subrelation. Still, the following result shows that in the consistent case both generate the same expanded belief set:

Theorem 6.8. *If $\vdash^T$ is a Tarski subrelation of a Scott consequence relation $\vdash$, then, for any A, either $\boldsymbol{K}_\vdash + A$ is inconsistent, or $\boldsymbol{K}_\vdash + A$ coincides with $\boldsymbol{K}_{\vdash^T} + A$.*

Lemma 6.1 above immediately implies the following result showing that expansions preserve both groundedness and base generation.

Lemma 6.9. *If $\vdash$ is generated by a set of propositions $\mathbb{B}$, then $\vdash_{+A}$ is generated by the set $\mathbb{B}_A = \mathbb{B} \cup \{A \wedge B \mid B \in \mathbb{B}\}$.*

This result implies, in particular, that expansion of a base-generated consequence relation with respect to a proposition A amounts simply to adding A to the base. Thus, in such cases our notion of basic expansion coincides with the corresponding notion of expansion in common base-oriented approaches. It agrees also with the notion of insertion for flocks described in [5].

Our last result in this section shows a way of constructing the expanded consequence relation on the basis of a given set of sequents ("dependence rules") determining the source consequence relation. Notice that this task is less trivial than the corresponding task for contractions, since we need to *remove* certain sequents from a consequence relation, rather than add some.

Theorem 6.10. *If $\vdash$ is a consequence relation, $\vdash_{+A}$ is a least consequence relation $\vdash_*$ satisfying the following two conditions*:

(1) *If $a \vdash c$, then $a \vdash_* c, A$.*
(2) *If $a \vdash c$ and $\neg A$ logically implies all propositions from a, then $a \vdash_* c$.*

The above theorem suggests the following method of constructing the expanded consequence relation. Let $\boldsymbol{R}$ be a set of generating sequents of $\vdash$. In order to construct a set $\boldsymbol{R}_A$ of generating sequents of $\vdash_{+A}$, we can do the following:

- For any sequent $a \vdash b$ from $\boldsymbol{R}$, add a sequent $a \vdash b, A$ to $\boldsymbol{R}_A$;
- Derive from $\boldsymbol{R}$ all minimal sequents $a \vdash b$ such that $\neg A$ logically implies all a's, and add these sequents to $\boldsymbol{R}_A$.

6.2. Conditional expansions

As we said, expansions can differ in the degree of firmness they assign to the newly added beliefs. In practice, additions of new beliefs are often conditional on other propositions we believe, e.g., on evidences we have or the reliability of witnesses that give us new information, etc. This idea can be captured using the notion of a conditional expansion introduced below.

Definition 6.3. Let A and B be propositions and $\mathcal{T}$ the set of all theories of a consequence relation $\vdash$. Then $\vdash_{+(A|B)}$ will denote a consequence relation determined by the set of theories

$$\mathcal{T}_{(A|B)} = \mathcal{T} \cup \{\mathrm{Th}(u \cup A) \mid u \in \mathcal{T} \text{ and } B \in u\}.$$

This consequence relation will be said to be obtained by a *conditional expansion* of $\vdash$.

As can be seen, a conditional addition of A relative to B is obtained by expanding each admissible belief state satisfying B with A. Basic expansion can be seen as a special case of conditional expansion when B is a tautology. The following theorem provides a direct description of a consequence relation resulting from a conditional expansion. Its proof can be obtained by a small change in the corresponding proof for basic expansions.

Theorem 6.11. *If $\vdash$ is a consequence relation, then*

$$a \vdash_{+(A|B)} b \quad \textit{iff} \quad a \vdash b \textit{ and } B, A \to a \vdash A \to b.$$

Conditional expansion can be seen as a qualitative counterpart of transmutation operations suggested in [29]. Just as the latter, it can raise the "degree of firmness" even for already believed propositions.

If the condition B is disbelieved in $\vdash$, the conditional expansion $+(A \mid B)$ will have no effect for it. But if B is believed, then A will also be believed in the resulting consequence relation. Moreover, if A was disbelieved previously, then the belief in A will depend on B in the resulting consequence relation, that is, we will have $A \vdash_{+(A|B)} B$. In other words, our willingness to believe in A in what follows will be conditional on acceptance of B. And this will remain to be so until A will receive some additional ground or justification.

6.3. Default expansions

Recall that a most simple and natural interpretation of non-determinate epistemic states consists in viewing them as generated by a certain set of defaults, or expectations. This suggests that reasonable expansions should also be capable of adding new defaults to our epistemic states.

Unlike basic propositions that are supposed to be freely conjoined to any admissible belief state, defaults are restricted in this respect by the principle of non-contradiction; we accept defaults only if there are no contradictory evidence. Consequently, we suggest to define a *default expansion* of an epistemic state $\mathcal{E}$ with respect to proposition A as obtained by adding A to all admissible states of $\mathcal{E}$ *that are consistent with* A.

For any set of theories $\mathcal{T}$, we will denote by $\mathcal{T}_{\delta A}$ the set of theories

$$\mathcal{T} \cup \big\{\mathrm{Th}(u \cup \{A\}) \mid u \in \mathcal{T} \ \&\ \neg A \notin u\big\}.$$

Definition 6.4. A *default expansion* of a consequence relation $\vdash$ with respect to a proposition A is a consequence relation, denoted by $\vdash_{\delta A}$, that is determined by the set of theories $\mathcal{T}_{\delta A}$, where $\mathcal{T}$ is a set of theories of $\vdash$.

Again, default expansions behave very similar to basic expansions. In fact, the only distinction between them is that if $\neg A$ belongs to some admissible belief state (that is, it is not disbelieved), then the basic expansion with respect to A will make the resulting epistemic state classically inconsistent. Default expansions, however, will always preserve classical consistency. Still, it is easy to see that any default expansion can be obtained as a combination of the corresponding basic expansion, followed by a contraction with respect to falsity:

$$\vdash_{\delta A} = \vdash_{\langle +A, -\bot\rangle}.$$

The proof for the following syntactic characterization of default expansions can be obtained by a slight modification of the proof for Theorem 6.4.

Theorem 6.12. *For any consequence relation $\vdash$ and any A,*

$$a \vdash_{\delta A} b \quad \textit{iff} \quad a \vdash b \textit{ and } A \to a \vdash A \to b, \neg A.$$

Notice that if $b \neq \emptyset$, then $A \to a \vdash A \to b, \neg A$ is reducible to $A \to a \vdash A \to b$, and hence $a \vdash_{\delta A} b$ will hold in this case if and only if $a \vdash_{+A} b$.

Just as for basic expansions, we have

Theorem 6.13. *$\vdash_{\delta A}$ is a greatest subrelation of $\vdash$ that makes A a default proposition.*

In addition, any default of $\vdash$ remains a default of $\vdash_{\delta A}$, so default expansions preserve default propositions. We have also that if a consequence relation is generated by a set of propositions $\mathbb{B}$, then $\vdash_{\delta A}$ is generated by $\mathbb{B}$ and all *consistent* conjunctions $A \wedge B$, where $B \in \mathbb{B}$. This implies, in particular, that default expansions will preserve groundedness and default generation.

7. Revisions

An operation of basic expansion may result in an epistemic state having an inconsistent belief set. To secure that addition of a set w to a consequence relation will produce only consistent theories, we need the following notion. An epistemic state $\mathcal{E}$ will be said to be *admissible* for adding a set of propositions w if every belief state from $\mathcal{E}$ is logically consistent with w. The following definition describes the corresponding notion for consequence relations.

Definition 7.1. A consequence relation $\vdash$ will be called *admissible for* (*adding*) a set of propositions w if, for any finite $a \subseteq w$, $\neg \bigwedge a \vdash$ holds.

As can be seen, in the case of adding a finite set a, the admissibility condition amounts simply to validity of a sequent $\neg \bigwedge a \vdash$. In other words, admissibility of a amounts to disbelieving $\neg \bigwedge a$. Clearly, if a consequence relation is admissible for some set of propositions, it is classically consistent, that is, the set of all formulas is not its theory. Moreover, we have

Lemma 7.1. *$\vdash$ is admissible for a set w iff $\vdash_{+w}$ is a classically consistent consequence relation.*

Thus, the admissibility of a consequence relation for some expansion amounts to disbelieving certain propositions. As was shown earlier (see Lemma 5.3), the latter is equivalent to a closure of the consequence relation with respect to a certain contraction. Consequently, we immediately obtain

Lemma 7.2. *A consequence relation $\vdash$ is admissible for a set w if and only if it coincides with $\vdash_{-w^{\neg}}$, where $w^{\neg} = \{\neg \bigwedge a \mid a \subseteq w\}$.*

Therefore, in order to make a consequence relation admissible, we need to perform first appropriate contraction producing disbelief in these propositions. This means, in particular, that a *classically consistent* expansion with respect to w is always equivalent to a sequence of changes $\langle -w^{\neg}, +w\rangle$. In other words, to produce consistent basic expansions, we can always employ a compound belief change operation $\langle -w^{\neg}, +w\rangle$ that is definable for all consequence relations.

A well-known principle, called *Levi identity* in [8], identifies a *revision* of a belief set with respect to a proposition A with a sequence of changes consisting of contracting $\neg A$ and subsequent expansion with A. Generalizing this principle to consequence relations, we introduce the following definition.

Definition 7.2. For any consequence relation $\vdash$ and any set of propositions w, we will denote by $\vdash_{*w}$ a consequence relation $\vdash_{\langle -w^{\neg}, +w\rangle}$. This consequence relation will be called a *basic revision* of $\vdash$ with respect to w.

As can be seen, revisions are definable for any consequence relation and always produce a classically consistent belief set. Moreover, if a consequence relation is admissible for adding a set w of propositions, than its revision with respect to w coincides with an expansion with respect to w. Thus, revisions coincide with expansions on admissible consequence relations. Furthermore, it is easy to see that basic expansions coincide with default expansions on admissible consequence relations. This means that default expansions could be safely used instead of basic expansions in the above definition of revisions. Thus, as far as revisions are concerned, the difference between these two kinds of expansions becomes inessential.

Recall that a contraction with respect to $\neg A$ always commutes with a basic expansion with respect to A. So, for a singular revision $*A$, the order of performing its underlying contraction and expansion does not matter; in each case we will obtain the same revised epistemic state. In other words, Levi's identity for revised epistemic states *can be reversed* (cf. [16]). Note, however, that this commutativity property is not extendable to multiple revisions.

Another interesting property of basic revisions is that, instead of basic expansions, we could safely use default expansions in their definition: as can be easily seen, $\vdash_{*w}$ is always equal to $\vdash_{\langle -w\neg,\delta w\rangle}$. In other words, the difference between basic and default expansions becomes inessential so far as we are only revising epistemic states.

As before, we will denote by $\boldsymbol{K}_{\vdash}*w$ the belief set of $\vdash_{*w}$. This set can be seen as a result of revising the belief set $\boldsymbol{K}_{\vdash}$ in order to accept w. Note, however, that Levi identity for revised consequence relations does *not* imply the more familiar *Levi identity for belief sets*:

$$\boldsymbol{K}_{\vdash}*A = \mathrm{Th}\big(\{A\} \cup (\boldsymbol{K}_{\vdash} - \neg A)\big).$$

The reasons for possible violations of this principle lie in the fact that it takes from the intermediate contracted epistemic state only its belief set and thereby "forgets" the actual alternatives that produced the latter.

The following counterexample involves a finite epistemic state that is both determinate and grounded.

Example 7.1. Let $\vdash$ be generated by the set of three propositions $\{q, (p \to q) \wedge r, \neg p \wedge q \wedge r\}$ with respect to the classical entailment. Its belief set $\boldsymbol{K}_{\vdash}$ will be $\mathrm{Th}_c(\neg p \wedge q \wedge r)$. Then, the revision of $\vdash$ with respect to p will produce the following generating set:

$$\{q, (p \to q) \wedge r, p \wedge q, p \wedge q \wedge r\}.$$

The associated revised consequence relation will again be determinate with $\mathrm{Th}_c(p \wedge q \wedge r)$ as its greatest theory. Consequently, the latter will constitute the revised belief set $\boldsymbol{K}_{\vdash}*p$. However, $\boldsymbol{K}_{\vdash} - \neg p$ is equal in our case to $\mathrm{Th}_c(q \vee (\neg p \wedge r))$, and consequently $\mathrm{Th}(\{p\} \cup (\boldsymbol{K}_{\vdash} - \neg p))$ will coincide with the smaller set $\mathrm{Th}(p \wedge q)$.

The above example provides a further evidence for our claim that forgetting the actual alternatives and replacing them by their intersections, as is done in common approaches to belief change, may lead to a loss of information. Still, an interesting (and somewhat surprising) fact is that revisions of base-generated epistemic states turn out to satisfy Levi identity for belief sets.

Theorem 7.3. *If $\vdash$ is finite and base-generated, then, for any A,*

$$\boldsymbol{K}_{\vdash}*A = \mathrm{Th}\big(\{A\} \cup (\boldsymbol{K}_{\vdash} - \neg A)\big).$$

As can be expected, revisions do not commute: the sequence of revisions $\langle *A, *\neg A\rangle$ produces a belief in $\neg A$ and hence does not coincide with $\langle *\neg A, *A\rangle$ that leads to a belief in A. Moreover, unlike contractions and expansions, multiple revisions are not always simulated by singular ones (though we always have $\vdash_{\langle *A,*B\rangle} \subseteq \vdash_{*\{A,B\}}$ – see the proof of the next theorem). Still, the following result gives necessary and sufficient conditions for both commutativity and reducibility of multiple revisions.

Theorem 7.4. *The following conditions are equivalent for any consequence relation* $\vdash$ *and any propositions* A *and* B:

(a) $\neg A \vee \neg B \vdash \neg A, \neg B$;
(b) $\vdash_{\langle *B,*A\rangle} = \vdash_{\langle *A,*B\rangle}$;
(c) $\vdash_{*\{A,B\}} = \vdash_{\langle *A,*B\rangle}$;
(d) $\vdash_{\langle *B,*A\rangle} \subseteq \vdash_{\langle *A,*B\rangle}$.

The condition (a) above says, in effect, that $\neg(A \wedge B)$ is a *derived* proposition that is admissible in the epistemic state only when either $\neg A$ or $\neg B$ is an admissible belief. The above result then shows that in this case the revisions with respect to A and B are compatible and do not "interfere" with each other; consequently, they can be performed in any order, as well as simultaneously.

The so-called *Harper* (or *Gärdenfors*) *identity* (see [8]) states that contraction of a belief set $\boldsymbol{K}$ with respect to A is precisely the common part of $\boldsymbol{K}$ and its revision with respect to $\neg A$, that is,

$$\boldsymbol{K} - A = (\boldsymbol{K} * \neg A) \cap \boldsymbol{K}.$$

This identity also does not hold for our belief operations. To see this, it is sufficient to note that it implies recovery for contractions that is not valid in our framework. Still, the following result shows that the corresponding "generalized" Gärdenfors identity holds for epistemic states: if $\mathcal{E}$ is an epistemic state, $\mathcal{E} - A$ its contraction with respect to A and $\mathcal{E} * \neg A$ its revision with respect to $\neg A$, then

$$\mathcal{E} - A = \mathcal{E} \cap \mathcal{E} * \neg A.$$

In other words, the contracted epistemic state contains exactly the admissible belief states that are common to both the source epistemic state and its revision with respect to $\neg A$. The following theorem gives a syntactic description of this result:

Theorem 7.5. $\vdash_{-\neg A}$ *is a least consequence relation containing* $\vdash$ *and* $\vdash_{*A}$.

7.1. Accessible consequence relations

Finally, let us consider the question what epistemic states and consequence relations can be generated from some "initial" ones by applying singular contractions and basic expansions.

Our distinction between belief and knowledge suggests that a starting point of belief change, a "tabula rasa" of beliefs, can be represented by an epistemic state containing a single admissible belief state. We will call such epistemic states *unitary* ones. All that is believed in such an epistemic state coincides with what is known in it. In other words, such states do not involve “genuine” beliefs that are distinct from knowledge. [4] Our belief change operations, however, can transform such states into ones involving nontrivial beliefs. Accordingly, any epistemic state (or a consequence relation) that can be obtained from some unitary one by applying singular contractions and expansions will be called *accessible*.

Since both contractions and expansions preserve groundedness and primeness, we immediately obtain that any accessible epistemic state is finite, grounded and prime. Moreover, the following result shows that any such epistemic state will be accessible.

Theorem 7.6. *An epistemic state is accessible iff it is finite, prime and grounded.*

The above result demonstrates that grounded epistemic states and consequence relations can be seen as an exact representation framework for finite belief change operations on epistemic states.

It is interesting to note that using only basic revisions as belief change operations, we cannot “access” in this way all grounded epistemic states:

Example 7.2. Let us consider an epistemic state generated by a set of propositions $\{\top, p, \neg p\}$ with respect to Th_c. To see that it cannot be produced by a sequence of basic revisions, note that theories $\mathrm{Th}_c(p)$ and $\mathrm{Th}_c(\neg p)$ could only be created by a direct revision with respect to p and $\neg p$, respectively, while each of these revisions removes the effect of the other.

So, belief change cannot be reduced to belief revision in the narrow sense.

7.2. Belief revision functions

Revisions of an epistemic state generate a corresponding *belief revision function* on its belief set. As for contractions and expansions, such belief functions are determined already by associated Tarski consequence relations:

Theorem 7.7. *If* $\vdash^T$ *is a Tarski subrelation of a Scott consequence relation* $\vdash$*, then* $\vdash$ *and* $\vdash^T$ *determine the same belief revision function.*

Thus, all belief change functions described in the paper are representable using ordinary Tarski consequence relations.

A detailed description of belief revision functions generated by epistemic states is given in [3]. In particular, we establish a close connection between them and nonmonotonic preferential inference relations from [19]. Thus, belief revision functions generated by epistemic states can be seen as a natural generalization of AGM revisions. In particular, they extend the scope of the correspondence between belief revision and nonmonotonic inference from rational inference relations to a broader class of preferential ones. Moreover, in order to satisfy all the AGM postulates for revision, we need only to

[4] Compare this with Gärdenfors’ identification of knowledge with full belief in [8].

restrict the class of generating consequence relations. This will demonstrate that the AGM "rationality" is not necessarily tied to the coherentist paradigm, but can also be reconstructed on a purely foundationalist basis.

8. Conclusions

We have argued in Section 1 that in order to provide a more adequate and uniform representation of belief change, we need a notion of an epistemic state more complex than simply a set of beliefs. Such an epistemic state should embody information about various dependencies among believed propositions that would be sufficient for determining the results of its reasonable revision in response to new data. In other words, a comprehensive representation of epistemic states ought to be sufficiently informative for determining the results of all subsequent changes made to it. In this respect, it seems that the suggested identification of epistemic states with sets of belief states, as well as their representation in terms of Scott consequence relations that encode such dependencies, provide a minimum of complexity required for this purpose.

The results and constructions described in the paper show that epistemic states provide a constructive representation framework that subsumes the two major approaches to belief change. Thus, we have shown that the base change paradigm is representable in our approach using some special class of epistemic states and consequence relations. In an accompanying paper [3] we show that the AGM theory can also be represented in our framework by viewing epistemic entrenchment as another special case of a Scott consequence relation. This can be seen as a step towards the unification of the two principal belief revision paradigms. We have seen, however, that the framework of epistemic states admits reasonable kinds of belief change operations that go far beyond what is strictly necessary for representing the two "traditional" theories of belief change. Our notion of an epistemic state provides a powerful generalization extending significantly the expressive capabilities of current approaches. Some of the possibilities have been discussed in the present paper. Further details can be found in [3] where we compare our framework with other constructions made in the AGM tradition.

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Appendix A. Proofs of the main results

Theorem 4.3. *A supraclassical Scott consequence relation* $\vdash$ *is grounded iff it is generated by a set of propositions with respect to some classical consequence relation.*

Proof. If $\vdash$ is grounded, let us take $\mathbb{B}$ to be the set of ground propositions of $\vdash$ and $\mathrm{Th}_{\vdash}$ to be an appropriate classical consequence relation. Then the set $\{\mathrm{Th}_{\vdash}(B) \mid B \in \mathbb{B}\}$ is clearly a set of theories generating $\vdash$.

In the other direction, we need only to prove that if $\vdash$ is generated by a pair $(\mathbb{B}, \mathrm{Th})$, any theory from the (generating) set $\{\mathrm{Th}(B) \mid B \in \mathbb{B}\}$ is a ground theory of $\vdash$. To this end, we will prove that, for any $B \in \mathbb{B}$, $\mathrm{Th}(B) = \mathrm{Th}_{\vdash}(B)$. Indeed, if $A \in \mathrm{Th}(B)$, then $B \to A \in \mathrm{Th}(B_i)$, for any $B_i \in \mathbb{B}$, and hence $\vdash B \to A$, that is, $A \in \mathrm{Th}_{\vdash}(B)$. In the other direction, if $\vdash B \to A$, then $B \vdash A$. But $\mathrm{Th}(B)$ is a theory of $\vdash$ and $B \in \mathrm{Th}(B)$. Consequently, we obtain $A \in \mathrm{Th}(B)$. $\square$

Theorem 4.4. *A consequence relation is flock-generated iff it is prime and grounded.*

Proof. Let $\mathcal{F}$ be a flock generating $\vdash$. Since $\mathrm{Th}(\emptyset)$ is a theory, $\vdash$ is a prime consequence relation. Due to compactness of Th, finite subsets of sets from $\mathcal{F}$ are sufficient for determining the generating consequence relation. Moreover, such finite sets can be always replaced by their conjunctions. In accordance with this, we will denote by $\mathbb{B}_{\mathcal{F}}$ the set of propositions $\bigcup\{u^{\wedge} \mid u \in \mathcal{F}\}$. Then it is easy to check that $\vdash$ is generated by $\mathbb{B}_{\mathcal{F}}$ relative to the same classical consequence relation.

In the other direction, if $\mathbb{B}$ is a set of propositions generating a prime consequence relation $\vdash$ with respect to Th, let us denote by $\mathcal{F}_{\mathbb{B}}$ the flock containing all sets $\{B\}$, where $B \in \mathbb{B}$. Let B_0 be a proposition generating the least theory of $\vdash$. We will denote by Th^0 a classical consequence relation defined as $\mathrm{Th}^0(a) = \mathrm{Th}(a, B_0)$. Then $\mathcal{F}_{\mathbb{B}}$ will be a flock generating $\vdash$ with respect to Th^0. $\square$

Theorem 4.6. *A is a basic proposition of $\vdash$ iff, for any theory u of $\vdash$, $\mathrm{Th}_c(\{A\} \cup u)$ is also a theory of $\vdash$.*

Proof. Let u be a theory of $\vdash$. If $\mathrm{Th}_c(\{A\} \cup u)$ is not a theory of $\vdash$, there must exist finite sets $a \subseteq u$, $c \subseteq \overline{\mathrm{Th}_c(\{A\} \cup u)}$ such that $a, A \vdash c$. If A is a basic proposition, we have $a \vdash A \to c$, and since u is a theory, there is $C \in c$ such that $A \to C \in u$. Consequently, $C \in \mathrm{Th}_c(\{A\} \cup u)$, contrary to the assumption that c is disjoint from $\mathrm{Th}_c(\{A\} \cup u)$.

In the other direction, if $a \nvdash A \to c$, then there is a theory u such that $a \subseteq u$ and $A \to c \subseteq \overline{u}$. But then $\mathrm{Th}_c(\{A\} \cup u)$ is a theory of $\vdash$ that includes A and a and is disjoint from c. Therefore, $A, a \nvdash c$, and hence, A is a basic proposition. $\square$

Lemma 4.8. *If $\mathbb{B}$ is a base of $\vdash$, then any proposition from $\mathbb{B}^{\wedge}$ is a basic proposition of $\vdash$.*

Proof. Let $\vdash$ be a consequence relation generated by a base $\mathbb{B}$ with respect to Th, and $B \in \mathbb{B}^{\wedge}$. Now, if $a \nvdash B \to c$, there must exist $B_1 \in \mathbb{B}^{\wedge}$ such that $a \subseteq \mathrm{Th}(B_1)$ and $B \to c \subseteq \overline{\mathrm{Th}(B_1)}$. The later condition implies that c is disjoint from $\mathrm{Th}(B \wedge B_1)$. In addition, $\mathrm{Th}(B \wedge B_1)$ includes both B and a. But $B \wedge B_1$ belongs to $\mathbb{B}^{\wedge}$, and hence $a, B \nvdash c$. Therefore, B is a basic proposition of $\vdash$. $\square$

Theorem 4.9. *A is a basic proposition of a base-generated consequence relation $\vdash$ iff A is ground in $\vdash$.*

Proof. Since Th(∅) is a theory of a base-generated consequence relation $\vdash$, if B is a basic proposition of $\vdash$, Th(B) is also a theory of $\vdash$, and hence, B is ground in $\vdash$.

Let $\vdash$ be a consequence relation generated by a base $\mathbb{B}$ with respect to Th, and assume that A is ground, that is, $\mathrm{Th}_{\vdash}(A)$ is a theory of $\vdash$. If $a \nvdash A \to c$, there must exist $B \in \mathbb{B}^{\wedge}$ such that $a \subseteq \mathrm{Th}(B)$ and $A \to c \subseteq \overline{\mathrm{Th}(B)}$. But B is a ground proposition of $\vdash$, and hence $\mathrm{Th}(B) = \mathrm{Th}_{\vdash}(B)$ (cf. the proof of Theorem 4.3). Consequently, we have $B \to c \subseteq \overline{\mathrm{Th}_{\vdash}(A)}$, and hence $A \nvdash B \to c$ (since $\mathrm{Th}_{\vdash}(A)$ is a theory). Therefore there must exist $B_1 \in \mathbb{B}^{\wedge}$ such that $A \in \mathrm{Th}(B_1)$ and $B \to c \subseteq \overline{\mathrm{Th}(B_1)}$. The later condition implies that c is disjoint from $\mathrm{Th}(B \wedge B_1)$. In addition, $\mathrm{Th}(B \wedge B_1)$ includes both A and a. But $B \wedge B_1$ belongs to $\mathbb{B}^{\wedge}$, and hence $a, A \nvdash c$. Therefore, A is a basic proposition of $\vdash$. □

Lemma 5.1. *For any finite sets a and b, $a \vdash_{-w} b$ iff $a \vdash b, w$.*

Proof. It is easy to check that a consequence relation $\vdash^*$ defined as follows:

$$a \vdash^* b \quad \equiv \quad a \vdash b, w$$

is a supraclassical Scott consequence relation containing $\vdash$. Moreover, $C \vdash^*$, for any $C \in w$. Since $\vdash_{-w}$ is a least such consequence relation, we immediately obtain that $\vdash \subseteq \vdash^*$.

If $a \vdash b, w$, then $a \vdash_{-w} b, w$, and hence $a \vdash_{-w} b, c$, for some finite $c \subseteq w$. But $C \vdash_{-w}$, for any $C \in c$. Hence, applying Cut, we obtain $a \vdash_{-w} b$. Thus, $\vdash_{-w}$ coincides with $\vdash^*$. □

Theorem 5.2. *If a consequence relation $\vdash$ is generated by a set of theories $\mathcal{T}$, $\vdash_{-w}$ is generated by the set of all theories from $\mathcal{T}$ that are disjoint from w.*

Proof. Let $\mathcal{T}_w$ be a set of theories from $\mathcal{T}$ that are disjoint from w. Then $C \vdash_{\mathcal{T}_w}$ holds for any $C \in w$. Hence $\vdash_{-w}$ is included in $\vdash_{\mathcal{T}_w}$.

If $a \nvdash_{-w} b$, then $a \nvdash b, w$, and hence there is a theory u from $\mathcal{T}$ that includes a and disjoint from both b and w. Consequently, u belongs to $\mathcal{T}_w$, and therefore $a \nvdash_{\mathcal{T}_w} b$. Thus, $\vdash_{-w}$ coincides with $\vdash_{\mathcal{T}_w}$. □

Lemma 5.3. *A proposition A is disbelieved in $\vdash$ iff $\vdash_{-A}$ coincides with $\vdash$.*

Proof. If $A \vdash$ holds, then it is easy to see that $a \vdash b, A$ holds iff $a \vdash b$ holds. In the other direction, if the latter equivalence holds, we have, in particular, that $A \vdash A$ should imply $A \vdash$, and hence A is disbelieved in $\vdash$. □

Theorem 5.4. $\vdash_{\langle -A, -B\rangle} = \vdash_{-\{A,B\}}$.

Proof. By Lemma 5.1, $a \vdash_{\langle -A, -B\rangle} b$ holds iff $a \vdash_{-A} b, B$ iff $a \vdash b, A, B$. By the same lemma, the latter amounts also to $a \vdash_{-\{A,B\}} b$. Hence the result. □

Lemma 5.5.

(1) $\vdash_{-(A\wedge B)} = \vdash_{-A} \bigcap \vdash_{-B}$;

(2) $\vdash_{-\{A,B\}}$ *is a least consequence relation containing both* $\vdash_{-A}$ *and* $\vdash_{-B}$.

Proof. (1) Immediate from the fact that $a \vdash b, A \wedge B$ holds in a supraclassical consequence relation if and only if $a \vdash b, A$ and $a \vdash b, B$.

(2) Clearly, $\vdash_{-\{A,B\}}$ includes both $\vdash_{-A}$ and $\vdash_{-B}$. In addition, if $\vdash^*$ is any consequence relation that includes the latter two, it includes also $\vdash$ and contains both $A \vdash^*$ and $B \vdash^*$. But then $\vdash_{-\{A,B\}}$ is included in $\vdash^*$, since it is defined as a least consequence relation satisfying these conditions. □

Lemma 5.6. *If A is disbelieved in $\vdash_{-B}$, then $\vdash_{-(A\wedge B)}$ coincides with $\vdash_{-A}$.*

Proof. $A \vdash_{-B}$ holds iff $A \vdash B$. The latter implies $A \vdash A \wedge B$, and hence $A \wedge B$ is provably equivalent to A in $\vdash$. But then $a \vdash b, A \wedge B$ holds iff $a \vdash b, A$. This concludes the proof. □

Lemma 5.7. *Let $\vdash^T$ be a Tarski subrelation of a Scott consequence relation $\vdash$. Then, for any proposition A, the belief set of $\vdash_{-A}$ coincides with the belief set of $\vdash^T_{-A}$.*

Proof. We will show that $\vdash$ and $\vdash^T$ have the same maximal theories that do not include A. Clearly, if u is a maximal theory of $\vdash$ that does not include A, then it is also a maximal such theory in $\vdash^T$. Assume now that u is a maximal theory of $\vdash^T$ that does not include A. Since $u \nvdash^T A$, we have also $u \nvdash A$, and hence there is a maximal theory u_0 of $\vdash$ that does not include A and such that $u \subseteq u_0$. Since u_0 is also a theory of $\vdash^T$, we have that u coincides with u_0. □

Theorem 5.8. *Two consequence relations are contraction-equivalent only if they coincide.*

Proof. Let $\vdash^1$ and $\vdash^2$ be two contraction-equivalent consequence relations. If they are distinct, they must have distinct theories. Assume that u is a theory of $\vdash^1$, but not of $\vdash^2$. We contract both these consequence relations with a set $\overline{u}$ of all propositions that do not belong to u. Clearly, u is still a theory and, moreover, a greatest theory of $\vdash^1_{-\overline{u}}$, and hence it coincides with its belief set. Assume that u is also a belief set of $\vdash^2_{-\overline{u}}$. Since any belief set is an intersection of maximal theories, and $\vdash^2_{-\overline{u}}$ does not have theories above u, the latter should be a theory of $\vdash^2_{-\overline{u}}$, which is impossible, since u is not a theory of $\vdash^2$. □

Lemma 5.9. *For any supraclassical consequence relation, $a \vdash b$ iff contraction of $\vdash$ with respect to $b \cup \{\bigwedge a\}$ coincides with contraction of $\vdash$ with respect to b.*

Proof. For any sets c, d, $c \vdash_{-(b\cup\{\bigwedge a\})} d$ holds if and only if $c \vdash d, b, \bigwedge a$. Similarly, $c \vdash_{-b} d$ amounts to $c \vdash d, b$. Now, if $a \vdash b$, these two sequents are equivalent, and this gives the direction from left to right. In addition, if $\vdash_{-(b\cup\{\bigwedge a\})}$ coincides with $\vdash_{-b}$, then, in particular, $a \vdash_{-(b\cup\{\bigwedge a\})}$ should imply $a \vdash_{-b}$. But the former sequent belongs to $\vdash_{-(b\cup\{\bigwedge a\})}$, and hence $a \vdash_{-b}$ holds, which is reducible to $a \vdash b$. □

Theorem 5.10. *For any epistemic state $\mathcal{E} \in \mathbb{E}$, $\vdash^{\mathcal{E}}$ is a supraclassical consequence relation. Moreover, for any proposition A, $\vdash^{\mathcal{E}]A[}$ coincides with the contraction of $\vdash^{\mathcal{E}}$ with respect to A.*

Proof. We show first that $\vdash^{\mathcal{E}}$ is a supraclassical consequence relation.

Supraclassicality. Assume that $A \vDash B$. We have $\mathcal{E}]B, B, A[= \mathcal{E}]B, A[$ by Idempotence, and hence $\mathcal{E}]B, A \wedge B[= \mathcal{E}]B, B[$ by Inversion. But $A \wedge B$ is logically equivalent to A, and hence $\mathcal{E}]B, A[= \mathcal{E}]B[$ by Equivalence and Idempotence. The latter is equivalent to $A \vdash^{\mathcal{E}} B$, and we are done.

Monotonicity. To show Monotonicity, it is sufficient to demonstrate that an arbitrary proposition can be always added to the premises and conclusions of a valid sequent. Now, $\mathcal{E}]b \cup \{\wedge a\}[= \mathcal{E}]b[$ implies $\mathcal{E}]b \cup \{A\} \cup \{\wedge a\}[= \mathcal{E}]b \cup \{A\}[$ by Commutativity, and hence any proposition can be added to the conclusions. In addition, if $\mathcal{E}]B[= \mathcal{E}$, then $\mathcal{E}]B, A[= \mathcal{E}]A[$, and hence by Inversion $\mathcal{E}]A \wedge B[= \mathcal{E}]B[$, that is, $\mathcal{E}]A \wedge B[= \mathcal{E}$. Consequently, a proposition A can also be always added to the premises of a valid sequent.

Cut. If $\mathcal{E}]B, A[= \mathcal{E}]B[$ and $\mathcal{E}]A \wedge B[= \mathcal{E}$, then $\mathcal{E}]A \wedge B[= \mathcal{E}]A[$ by Inversion from the first equality, and hence $\mathcal{E}]A[= \mathcal{E}$. This shows the validity of the Cut rule.

Finally, $a \vdash^{\mathcal{E}]A[} b$ amounts to $\mathcal{E}]\{A\} \cup b \cup \{\wedge a\}[= \mathcal{E}]\{A\} \cup b[$. By Commutativity, the latter is equivalent to $\mathcal{E}]b \cup \{A\} \cup \{\wedge a\}[= \mathcal{E}]b \cup \{A\}[$, which is translatable as $a \vdash^{\mathcal{E}} b, A$. But the latter is equivalent to $a \vdash^{\mathcal{E}}_{-A} b$. This concludes the proof of the theorem. □

Theorem 5.12. *If $\vdash$ is a base-generated consequence relation, then $A \vdash C$ holds iff, for any proposition B, $A \in \boldsymbol{K}_{\vdash} - B$ implies $C \in \boldsymbol{K}_{\vdash} - B$.*

Proof. The direction from left to right is immediate. Assume that $A \nvdash C$. Then there must exist a base proposition B_0 such that $A \in \mathrm{Th}_{\vdash}(B_0)$ and $C \notin \mathrm{Th}_{\vdash}(B_0)$. Let us denote by B a proposition $B_0 \to C$. Then B_0 belongs to $\boldsymbol{K} - B$ (see Lemma 5.14), and hence A also belongs to $\boldsymbol{K} - B$. In addition, $C \notin \boldsymbol{K} - B$. Indeed, otherwise B would belong to $\boldsymbol{K} - B$, and hence $\vdash B_0 \to C$ that contradicts the assumption that $C \notin \mathrm{Th}_{\vdash}(B_0)$. Thus, the implication from right to left also holds. □

Theorem 5.13. *A basic proposition B of a consequence relation $\vdash$ will also be basic in $\vdash_{-w}$ iff $B \in \boldsymbol{K}_{\vdash} - w$.*

Proof. We will prove first the following auxiliary result:

Auxiliary Lemma. *Let B be a basic proposition of $\vdash$. Then B will also be a basic proposition of $\vdash_{-w}$ iff $B \to A \vdash w$, for each $A \in w$.*

Proof. Assume that B is a basic proposition of $\vdash_{-w}$. $B, B \to A \vdash A$ is a valid sequent of $\vdash$, and hence also $B, B \to A \vdash w$, for each $A \in w$. The latter sequents are equivalent to $B, B \to A \vdash_{-w}$. Since B is a basic proposition of $\vdash_{-w}$, we conclude that $B \to A \vdash_{-w}$, that is, $B \to A \vdash w$.

In the other direction, if $a, B \vdash_{-w} b$, then $a, B \vdash w, b$, and hence $a, B \vdash c, b$, for some finite $c \subseteq w$. Since B is a basic proposition of $\vdash$, we obtain $a \vdash B \to c, B \to b$. But $B \to A \vdash A$, for any $A \in c$, and consequently we have $a \vdash c, B \to b$ and therefore $a \vdash w, B \to b$. But the latter sequent is equivalent to $a \vdash_{-w} B \to b$, and hence B is a basic proposition of $\vdash_{-w}$. □

Now if B is basic in $\vdash_{-w}$, it is clearly believed in $\vdash_{-w}$. Assume that B is believed in $\vdash_{-w}$ As in the proof of the auxiliary lemma, we have $B, B \to A \vdash_{-w}$, for any $A \in w$. By Lemma 4.1 we conclude $B \to A \vdash_{-w}$, and now the result follows from the auxiliary lemma. □

Lemma 5.14. *B is a basic proposition of $\vdash$ iff, for any c, B is believed in $\vdash_{-(B\to c)}$.*

Proof. Notice first that $a, B \vdash c$ is always equivalent to $a, B \vdash B \to c$. Consequently, B is a basic proposition if and only if $a, B \vdash B \to c$ always implies $a \vdash B \to c$. Now the result follows from Lemma 4.1. □

Lemma 6.1. *If $\vdash$ is generated by a set of theories $\mathcal{T}$, then $\vdash_{+w}$ is generated by the set $\mathcal{T}_{+w}$.*

Proof. Let $\mathcal{T}^{\vdash}$ be a set of all theories of $\vdash$. Since $\mathcal{T} \subseteq \mathcal{T}^{\vdash}$, $\mathcal{T}_{+w}$ is included in $\mathcal{T}^{\vdash}_{+w}$. Consequently, $\vdash_{+w}$ is included in $\vdash_{\mathcal{T}_{+w}}$.

If $a \nvdash_{+w} b$, then there exists a subset v of w and a theory u of $\vdash$ such that $a \subseteq \mathrm{Th}_c(v \cup u)$ and b is disjoint from $\mathrm{Th}_c(v \cup u)$. Due to compactness, v can be taken to be a finite set. Consequently, $\bigwedge v \to a$ is a subset of u and u is disjoint from $\bigwedge v \to c$. Therefore, $\bigwedge v \to a \nvdash \bigwedge v \to c$, and hence there exists a theory $u_0 \in \mathcal{T}$ that includes $\bigwedge v \to a$ and is disjoint from $\bigwedge v \to c$. But then $\mathrm{Th}_c(v \cup u_0)$ is a theory from $\mathcal{T}_{+w}$ that includes a and is disjoint from b, and hence $a \nvdash_{\mathcal{T}_{+w}} b$. Therefore, $\vdash_{+w}$ coincides with $\vdash_{\mathcal{T}_{+w}}$. □

Theorem 6.2. *$\vdash_{+w}$ is a greatest subrelation of $\vdash$ that makes all propositions from w basic ones.*

Proof. If $\mathcal{T}$ is a set of theories of $\vdash$, then $\mathcal{T}_{+w}$ is clearly the least extension of $\mathcal{T}$ satisfying the condition that, for any $A \in w$, if u belongs to it, then $\mathrm{Th}_c(u, A)$ also belongs to it. Consequently, $\mathcal{T}_{+w}$ is a least set of theories that makes all propositions from w basic, and hence any other consequence relation having this feature will be included in $\vdash_{+w}$. □

Theorem 6.4. *$a \vdash_{+A} b$ iff $a \vdash b$ and $A \to a \vdash A \to b$.*

Proof. We will show that the consequence relation $\vdash_*$ determined by the latter condition is generated by the set $\mathcal{T}_{+A}$, where $\mathcal{T}$ is a set of theories of $\vdash$.

By the Representation Theorem, $a \vdash b$ holds iff $b \cap u \neq \emptyset$, for any theory u of $\vdash$ such that $a \subseteq u$. Similarly, $A \to a \vdash A \to b$ holds iff $b \cap \mathrm{Th}_c(\{A\} \cup u) \neq \emptyset$, for any theory u of $\vdash$ such that $a \subseteq \mathrm{Th}_c(\{A\} \cup u)$. Therefore, if $\mathcal{T}$ is a set of theories of $\vdash$, both these sequents hold simultaneously iff $b \cap u_0 \neq \emptyset$, for any theory u_0 from $\mathcal{T}_{+A}$ such that $a \subseteq u_0$. But the latter is equivalent to $a \vdash_{+A} b$, and we are done. □

Theorem 6.5. *For any consequence relation, $\vdash_{+w\cup\{A\}} = \vdash_{\langle +w,+A\rangle}$.*

Proof. If $\mathcal{T}$ is a set of theories of $\vdash$, then it is easy to see that $\mathcal{T}_{+w\cup\{A\}}$ is representable as a union of $\mathcal{T}_{+w}$ and the set $\{\mathrm{Th}_c(u \cup \{A\}) \mid u \in \mathcal{T}_{+w}\}$. Hence, $\vdash_{+w\cup\{A\}}$ can be obtained by first adding w and then adding A. □

Theorem 6.6. *The set of all theories of* $\vdash_{+A}$ *coincides with* $\mathcal{T}_{+A}$.

Proof. Let u be a theory of $\vdash_{+A}$. We show first that if $u \notin \mathcal{T}$, then $A \in u$. Indeed, otherwise we would have $u \vdash \overline{u}$ and $A \in \overline{u}$, and consequently $a \vdash c, A$, for some $a \subseteq u$, $c \subseteq \overline{u}$. By the above theorem, the latter implies $a \vdash_{+A} c, A$ and hence $u \vdash_{+A} \overline{u}$—a contradiction with the fact that u is a theory of $\vdash_{+A}$.

Assume now that $A \in u$. Let us show that $A \to u \nvdash \overline{u}$. Indeed, otherwise there are $B \in u$ and $c \subseteq \overline{u}$ such that $A \to B \vdash c$. This implies $A \to B \vdash_{+A} c$ and consequently $u \vdash_{+A} \overline{u}$, contrary to the fact that u is a theory of $\vdash_{+A}$.

The above condition implies that there must exist a theory u_0 of $\vdash$ such that $A \to u \subseteq u_0 \subseteq u$. But if $B \in u$, for some proposition B, then $A \to B \in u_0$, and consequently $B \in \mathrm{Th}_c(\{A\} \cup u_0)$. Thus, u coincides with $\mathrm{Th}_c(\{A\} \cup u_0)$, and hence it belongs to $\mathcal{T}_{+A}$. □

Corollary 6.7. *Any basic proposition of* $\vdash$ *remains a basic proposition of* $\vdash_{+A}$.

Proof. Let B be a basic proposition of $\vdash$ and u a theory of $\vdash_{+A}$. If u is also a theory of $\vdash$, then $\mathrm{Th}_c(u, B)$ is a theory of $\vdash$, and hence it is a theory of $\vdash_{+A}$. Otherwise there is a theory of u' of $\vdash$ such that $u = \mathrm{Th}_c(u', A)$. But then $\mathrm{Th}_c(u', B)$ is a theory of $\vdash$, and hence $\mathrm{Th}_c(u, B) = \mathrm{Th}_c(A, \mathrm{Th}_c(u', B))$ is a theory of $\vdash_{+A}$. □

Theorem 6.8. *If* $\vdash^T$ *is a Tarski subrelation of a Scott consequence relation* $\vdash$, *then, for any* A, *either* $\boldsymbol{K}_{\vdash} + A$ *is inconsistent, or* $\boldsymbol{K}_{\vdash} + A$ *coincides with* $\boldsymbol{K}_{\vdash^T} + A$.

Proof. The "expanded" belief set $\boldsymbol{K}_{\vdash} + A$ is an intersection of all maximal theories of the form $\mathrm{Th}(A, u)$, where u is a maximal theory of $\vdash$. Hence the result follows from the fact that $\vdash$ and $\vdash^T$ have the same maximal consistent theories. □

Theorem 6.10. *If* $\vdash$ *is a consequence relation,* $\vdash_{+A}$ *is a least consequence relation* $\vdash_*$ *satisfying the following two conditions*:

(1) *If* $a \vdash c$, *then* $a \vdash_* c, A$.

(2) *If* $a \vdash c$ *and* $\neg A$ *logically implies all propositions from* a, *then* $a \vdash_* c$.

Proof. Let $\vdash_0$ be the least consequence relation satisfying the above conditions. Since $\vdash_{+A}$ satisfies these conditions (this follows immediately from Theorem 6.4), we obtain that $\vdash_0$ is included in $\vdash_{+A}$.

If $a \vdash_{+A} b$, then $a \vdash b$ and $A \to a \vdash A \to b$, and hence $a \vdash_0 b, A$ and $A \to a \vdash_0 A \to b$. The latter sequent implies $A, a \vdash_0 b$, and hence applying Cut we obtain $a \vdash_0 c$. Thus, $\vdash_0$ coincides with $\vdash_{+A}$. □

Lemma 7.1. $\vdash$ *is admissible for a set* w *iff* $\vdash_{+w}$ *is a classically consistent consequence relation.*

Proof. If $\vdash$ is not admissible for w, we have $\neg \bigwedge a \nvdash$, for some finite $a \subseteq w$, and hence there exists a theory u that includes $\neg \bigwedge a$. Let v be a maximal theory containing u. Then $\mathrm{Th}(v \cup w)$ is a classically inconsistent theory, and hence $\vdash_{+w}$ is not classically consistent. In the other direction, if $\vdash_{+w}$ is not classically consistent, one of the "newly added" theories $\mathrm{Th}(u \cup w)$ should be inconsistent. But then w must contain a finite subset a such that $u \vDash \neg \bigwedge a$. Since u is deductively closed, $\neg \bigwedge a$ belongs to u, and consequently $\neg \bigwedge a \nvdash$ holds. Thus, $\vdash$ is not admissible for w. □

Theorem 7.3. *If $\vdash$ is finite and base-generated, then, for any A,*

$$\boldsymbol{K}_{\vdash} * A = \mathrm{Th}\big(\{A\} \cup (\boldsymbol{K}_{\vdash} - \neg A)\big).$$

Proof. Let $\mathbb{B}$ be a finite base of $\vdash$. Since $\vdash$ is determinate, if A is consistent with $\mathbb{B}$, the result follows immediately from the fact that revision coincides in this case with expansion. So, let us assume that $\mathbb{B}$ implies $\neg A$. Let $\mathcal{B}_A$ denote the set of all maximal subsets of $\mathbb{B}$ that are consistent with A. Then the contracted belief set $\boldsymbol{K}_{\vdash} - \neg A$ is an intersection of all theories $\mathrm{Th}(b_i)$, for $b_i \in \mathcal{B}_A$, whereas $\boldsymbol{K} * A$ is an intersection of *maximal* theories among all theories of the form $\mathrm{Th}(A, b_i)$, where $b_i \in \mathcal{B}_A$. As can be easily seen, the above Levi identity will hold if all theories $\mathrm{Th}(A, b_i)$ are maximal, that is, when they are incomparable with respect to set inclusion. We will show that this is indeed the case. Suppose, on the contrary, that there are $b_1, b_2 \in \mathcal{B}_A$ such that $\mathrm{Th}(A, b_1) \subseteq \mathrm{Th}(A, b_2)$. Then $b_1 \subseteq \mathrm{Th}(A, b_2)$, and hence $A, b_2 \vDash \bigwedge b_1$. But $b_1 \cup b_2$ cannot already be consistent with A, and hence $b_1, b_2 \vDash \neg A$. Combining these two entailments, we immediately conclude $b_1 \vDash \neg A$, which is impossible, since b_1 is consistent with A. This concludes the proof. □

Theorem 7.4. *The following conditions are equivalent for any consequence relation $\vdash$ and any propositions A and B:*
- (a) $\neg A \vee \neg B \vdash \neg A, \neg B$;
- (b) $\vdash_{\langle *B, *A\rangle} = \vdash_{\langle *A, *B\rangle}$;
- (c) $\vdash_{*\{A,B\}} = \vdash_{\langle *A, *B\rangle}$;
- (d) $\vdash_{\langle *B, *A\rangle} \subseteq \vdash_{\langle *A, *B\rangle}$.

Proof. Using the definition of revisions in terms of contractions and expansions and "unpacking" the corresponding characterizations of the latter, $a \vdash_{*\{A,B\}} b$ can be shown to be equivalent to validity of the following four sequents:

$$a \vdash b, \neg A \vee \neg B \qquad A \to a \vdash A \to b, \neg A \vee \neg B$$
$$(A \wedge B) \to a \vdash (A \wedge B) \to b, \neg A \vee \neg B \qquad B \to a \vdash B \to b, \neg A \vee \neg B$$

while $a \vdash_{\langle *A, *B\rangle} b$ is equivalent to

$$a \vdash b, \neg A, \neg B \qquad A \to a \vdash A \to b, \neg A \vee \neg B$$
$$(A \wedge B) \to a \vdash (A \wedge B) \to b, \neg A \vee \neg B \qquad B \to a \vdash B \to b, \neg A, \neg B.$$

The above conditions imply that $\vdash_{\langle *A, *B\rangle}$ is always included in $\vdash_{*\{A,B\}}$, that (a) implies both (b) and (c) and that (b) implies (d). Now, substituting $\{\neg A\}$ for a and $\emptyset$ for b in the sequents for $a \vdash_{\langle *A, *B\rangle} b$ and $a \vdash_{\langle *B, *A\rangle} b$, we immediately obtain that (d) implies (a). Similarly, substituting $\{\neg A \vee \neg B\}$ for a and $\emptyset$ for b in the sequents for $a \vdash_{\langle *A, *B\rangle} b$ and $a \vdash_{\{*A, *B\}} b$, we obtain that (c) also implies (a). □

Theorem 7.5. $\vdash_{-\neg A}$ *is a least consequence relation containing* $\vdash$ *and* $\vdash_{*A}$.

Proof. Let $\vdash^*$ be the least consequence relation containing $\vdash$ and $\vdash_{*A}$. Clearly, $\vdash^*$ is included in $\vdash_{-\neg A}$, since the latter contains both. Now, we have $\neg A \vdash_{*A}$, and hence $\neg A \vdash^*$. But $\vdash_{-\neg A}$ is a least consequence relation containing $\vdash$ and $\neg A \vdash$. Thus, $\vdash^*$ coincides with $\vdash_{-\neg A}$. $\square$

Theorem 7.6. *An epistemic state is accessible iff it is finite, prime and grounded.*

Proof. (A sketch) The set $\mathbb{B}$ of generating propositions of an epistemic state is partially ordered by the relation of logical consequence. Hence, we can stratify $\mathbb{B}$ into a finite number of layers $\mathbb{B}_i$ determined by the length of a maximal down path in this order. The desired sequence of changes will then be constructed by induction on the layers as follows: If $\mathcal{E}_i$ is an epistemic state constructed at stage i, then we expand it by adding all propositions from the layer $\mathbb{B}_{i+1}$, and then contract all propositions of the form $B \wedge B_1$, where $B \in \mathbb{B}_{i+1}$ and B_1 is any proposition from the same or a lower layer that is not a logical consequence of B. It is easy to check that each $\mathcal{E}_i$ will be generated by all the propositions from $\mathbb{B}$ that belong to layers $\leqslant i$. Consequently, the final epistemic state obtained by this construction will coincide with $\mathcal{E}$. $\square$

Theorem 7.7. *If* $\vdash^T$ *is a Tarski subrelation of a Scott consequence relation* $\vdash$, *then* $\vdash$ *and* $\vdash^T$ *determine the same belief revision function.*

Proof. $\boldsymbol{K}*A$ is an intersection of maximal theories among all theories of the form $\mathrm{Th}(A, u)$, where u is some maximal theory of $\vdash$ that does not contain $\neg A$. But as is shown in the proof of Lemma 5.7, $\vdash$ and $\vdash^T$ have the same maximal theories of this kind. This concludes the proof. $\square$

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